q–IDENTITIES OF FU AND LASCOUX PROVED BY THE q–RICE FORMULA

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Abstract. Two recent q–identities of Fu and Lascoux are proved by the q–Rice formula.

1. Introduction

Fu and Lascoux [5] (answering questions of Corteel and Lovejoy, related to identities in [2]) proved the following two identities:

\[
\sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} (x + q^{i-1}) \frac{q^{mi}}{(1 - q^i)^m} = \sum_{i=1}^{n} (1 - (-x)^i) \frac{q^i}{1 - q^i} \sum_{i_2 \leq \cdots \leq i_m \leq n} \frac{q^{i_2}}{1 - q^{i_2}} \cdots \frac{q^{i_m}}{1 - q^{i_m}} \tag{1.1}
\]

and

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^{i-1} (x + q^{i-1}) \frac{t^i}{1 - t^i} = -\frac{(q; q)_n}{(t; q)_{n+1}} \sum_{i=0}^{n} \frac{(t; q)_i}{(q; q)_i} (-xq)^i. \tag{1.2}
\]

Here, we use the usual notation \((x; q)_n = (1 - x)(1 - xq)\cdots(1 - xq^{n-1})\) and \(\binom{n}{k} = (q; q)_n/(q; q)_k(q; q)_{n-k}\), see [1].

In this short note, we will provide alternative attractive proofs of these, using the q–Rice formula, see [6] for some background and applications. Another proof has been obtained recently by Zeng [7] (added during revision).

2. Proof of Identity (1.1)

The q–Rice formula [6] allows to write an alternating sum as a contour integral:

\[
\sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} q^{(i)} f(q^{-i}) = \frac{1}{2\pi i} \int_{C} \frac{(q; q)_n}{(z; q)_{n+1}} f(z) dz,
\]

where the curve \(C\) encircles the poles \(q^{-1}, \ldots, q^{-n}\) and no others. For more technical details, see [6]. Under mild conditions, the integral (and thus the sum) can be expressed as the negative sum of the further residues. Thus, the computation of the alternating sum boils down to a residue computation.
In our application, we must find \( f(z) \) such that
\[
\begin{align*}
\qquad f(q^{-i}) &= (x + 1) \cdots (x + q^{i-1}) \frac{q^{mi}}{(1 - q^i)^m} q^{-i}(z) \\
&= (1 + x) \cdots \left(1 + \frac{x}{q^i-1}\right) \frac{1}{(q^{-i} - 1)^m}.
\end{align*}
\]

Now
\[
\begin{align*}
(1 + x) \cdots \left(1 + \frac{x}{q^i-1}\right) &= \prod_{h \geq 1} \frac{1 + xq^h}{1 + xq^h},
\end{align*}
\]

and thus we take
\[
\begin{align*}
f(z) &= \frac{1}{(z - 1)^m} \prod_{h \geq 1} \frac{1 + xzq^h}{1 + xq^h}.
\end{align*}
\]
The only extra pole is at \( z = 1 \), and so the sum is given by
\[
\begin{align*}
\text{SUM} &= -\text{Res}_{z=1} \frac{(q; q)_n}{(z; q)_{n+1}} \frac{1}{(z - 1)^m} \prod_{h \geq 1} \frac{1 + xzq^h}{1 + xq^h} \\
&= -[(z - 1)^{-1}] \frac{(q; q)_n}{(z; q)_{n+1}} \frac{1}{(z - 1)^m} \prod_{h \geq 1} \frac{1 + xzq^h}{1 + xq^h} \\
&= [(z - 1)^m] \frac{(q; q)_n}{(z; q)_n} \prod_{h \geq 1} \frac{1 + xzq^h}{1 + xq^h} \\
&= [w^m] \frac{1}{(1 - w \frac{q}{1 - q}) \cdots (1 - w \frac{q^n}{1 - q^n})} \prod_{h \geq 1} \left(1 + \frac{xwq^h}{1 + xq^h}\right).
\end{align*}
\]

It is not hard to see that
\[
\prod_{h \geq 1} \left(1 + \frac{xwq^h}{1 + xq^h}\right) = 1 - w \sum_{i \geq 1} (-x)^i \frac{q^i}{1 - q^i} \prod_{1 \leq h < i} \left(1 - w \frac{q^h}{1 - q^h}\right).
\]

To sketch a proof, let us look at the coefficient of \( w^2 \):
\[
\sum_{1 \leq h_1 < h_2} \frac{xq^{h_1}}{1 + xq^{h_1}} \frac{xq^{h_2}}{1 + xq^{h_2}} = \sum_{1 \leq h_1 < h_2, k_1 \geq 1} \frac{xq^{h_1}}{1 + xq^{h_1}} (-1)^{k_1-1} x^{k_1} q^{h_2 k_1}
\]
\[
= \sum_{1 \leq h_1, k_1 \geq 1} q^{h_1(k_1+1)} (-1)^{k_1-1} x^{k_1+1} \frac{q^{k_1}}{1 - q^{k_1}}
\]
\[
= \sum_{1 \leq h_1, k_1 \geq 1, k_2 \geq 0} q^{h_1(k_1+1)} q^{h_1 k_2} (-1)^{k_1+k_2-1} x^{k_1+k_2+1} \frac{q^{k_1}}{1 - q^{k_1}}
\]
\[
= \sum_{1 \leq h_1, 1 \leq k_1 < k_2} q^{h_1 k_2} (-1)^{k_2} x^{k_2} \frac{q^{k_1}}{1 - q^{k_1}}
\]
\[
= \sum_{1 \leq k_1 < k_2} (-x)^{k_2} q^{k_1} \frac{q^{k_1}}{1 - q^{k_1}} \frac{q^{k_2}}{1 - q^{k_2}}.
\]
If one does this, say, also for the coefficient of \(w^3\), then one quickly discovers the general pattern, and these coefficients are the same as the coefficients of the right side.\(^1\)

Now
\[
[w^m] \frac{1}{(1 - w \frac{q}{1-q}) \cdots (1 - w \frac{q^n}{1-q^n})} = \sum_{i=1}^{n} \frac{q^i}{1-q^i} \sum_{i \leq i_2 \leq \cdots \leq i_m \leq n} \frac{q^{i_2}}{1-q^{i_2}} \cdots \frac{q^{i_m}}{1-q^{i_m}}
\]
is already known (Dilcher’s sum \([3, 6]\)), so we are left to prove that
\[
\sum_{i=1}^{n} (-x)^i \frac{q^i}{1-q^i} \sum_{i \leq i_2 \leq \cdots \leq i_m \leq n} \frac{q^{i_2}}{1-q^{i_2}} \cdots \frac{q^{i_m}}{1-q^{i_m}} = [w^m] \frac{1}{(1 - w \frac{q}{1-q}) \cdots (1 - w \frac{q^n}{1-q^n})} w \sum_{i \geq 1} (-x)^i \frac{q^i}{1-q^i} \prod_{1 \leq h < i} (1 - w^{q_h})
\]

In terms of generating functions, we should show that
\[
\sum_{i=1}^{\infty} (-x)^i w a_i \frac{1}{(1 - w a_i) \cdots (1 - w a_n)} = \frac{1}{(1 - w a_1) \cdots (1 - w a_n)} \sum_{i \geq 1} (-x)^i w a_i \prod_{1 \leq h < i} (1 - w a_h),
\]
where we wrote \(a_i = q^i/(1 - q^i)\) (but it holds in general). But this in equivalent to
\[
\sum_{i=1}^{\infty} (-x)^i w a_i \frac{(1 - w a_1) \cdots (1 - w a_n)}{(1 - w a_i) \cdots (1 - w a_n)} = \sum_{i \geq 1} (-x)^i w a_i \prod_{1 \leq h < i} (1 - w a_h),
\]
and thus proved.

3. Proof of Identity (1.2)

This time we take
\[
f(z) = \frac{1}{z - t} \prod_{h \geq 1} \frac{1 + xz q^h}{1 + x q^h}
\]
and write
\[
\text{SUM} = \frac{1}{2\pi i} \int_{C} \frac{(z; q)_n}{(z; q)_{n+1}} \frac{1}{z - t} \prod_{h \geq 1} \frac{1 + xz q^h}{1 + x q^h} dz.
\]
Now we use the \(q\)-binomial theorem (sometimes called Cauchy’s formula):
\[
\prod_{h \geq 1} \frac{1 + xz q^h}{1 + x q^h} = \frac{(-xzq; q)_\infty}{(-xq; q)_\infty} = \sum_{m \geq 0} \frac{(z; q)_m}{(q; q)_m} (-xq)^m.
\]

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\(^1\)Robin Chapman (private communication) has provided a simple combinatorial proof by interpreting both sides as \(\sum_{\pi} (-x)^{\text{number of parts of } \pi} (-w)^{\text{number of distinct parts of } \pi_q|\pi|}\).
However, for the residues at $z = q^{-i}$, $i = 0, \ldots, n$, only the terms for $m \leq n$ are relevant. Henceforth we may write

$$\sum = \frac{1}{2\pi i} \int_{C} \frac{(q; q)_n}{(z; q)_{n+1}} \frac{1}{z - t} \sum_{m=0}^{n} (z; q)_m (-xq)^m dz.$$  

For outside residues, there is only one, at $z = t$, and therefore

$$\sum = -\text{Res}_{z=t} \frac{(q; q)_n}{(z; q)_{n+1}} \frac{1}{z - t} \sum_{m=0}^{n} (z; q)_m (-xq)^m$$

$$= -\frac{(q; q)_n}{(t; q)_{n+1}} \sum_{m=0}^{n} (t; q)_m (-xq)^m.$$  

This is clearly equivalent to the formula of Fu and Lascoux.

4. CONCLUSION

This method works equally well for similar sums, like

$$\sum = \sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right] (-1)^{i-1} (x + 1) \ldots (x + q^{i-1}) \frac{q^i}{(1 - tq^i)^2},$$  

with the result

$$\sum = -\text{Res}_{z=t} \frac{(q; q)_n}{(z; q)_{n+1}} \frac{z}{(z - t)^2} \sum_{m=0}^{n} (z; q)_m (-xq)^m.$$  

Rice’s formula belongs to the realm of divided differences, see [4]. This is what links our method and the one of Fu and Lascoux.

REFERENCES