Simons [2] has proved the identity
\[ \sum_{q=0}^q (-1)^{q+r}(q+r)!(1+x)^r = \sum_{r=0}^q (q+r)!x^r; \quad (1) \]
Chapman [1] gave a nice and short proof of it. In this note, I want to give another attractive proof. It uses Cauchy’s integral formula to pull out coefficients of generating functions.

We divide (1) by \( q! \) and prove the equivalent version
\[ S = \sum_{r=0}^q \binom{q}{r} \binom{q+r}{r} (-1)^{q+r}(1+x)^r = \sum_{r=0}^q \binom{q+r}{r} x^r. \]

We start with the righthand-side:
\[ S = \left[t^q\right] \sum_{i \geq 0} \binom{q}{i} t^i \cdot \sum_{i \geq 0} \binom{q+i}{i} (tx)^i \]
\[ = \left[t^q\right] (1+t)^q \cdot (1-tx)^{-q-1} \]
\[ = \frac{1}{2\pi i} \oint \frac{dt}{t^{q+1}} (1+t)^q \cdot (1-tx)^{-q-1}. \]

Now we substitute \( t = u/(1-u) \), so that \( dt = du/(1-u)^2 \) and obtain
\[ S = \frac{1}{2\pi i} \oint \frac{du}{(1-u)^2} \frac{(1-u)^{q+1}}{u^{q+1}} (1-u)^{-q} \cdot \left(1 - u(1+x) \right)^{-q-1} \]
\[ = [u^q](1-u)^q(1-u(1+x))^{-q-1} \]
\[ = \sum_{r=0}^q \binom{-q-1}{r} (-1)^r(1+x)^r \binom{q}{q-r} (-1)^{q-r} \]
\[ = \sum_{r=0}^q \binom{q+r}{r} \binom{q}{r} (1+x)^r(-1)^{q-r}, \]
which is the lefthand-side.

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References


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