INFINITE 0-1-SEQUENCES WITHOUT LONG
ADJACENT IDENTICAL BLOCKS

Helmut PRODINGER and Friedrich J. URBANEK
Institut für Mathematische Logik und Formale Sprachen, Technische Universität, Wien, Austria

Received 5 January 1978
Revised 17 May 1979

This paper deals with sequences \(a_1a_2a_3 \cdots\) of symbols 0 and 1 with the property that they contain no arbitrary long blocks of the form \(a_{i+1} \cdots a_{i+k} = \text{ww} \). The behaviour of this class of sequences with respect to some operations is examined. Especially the following is shown: Let be \(a_i^{(0)} = a_n, a_i^{(n+1)} = (1/i) \sum_{k=1}^{n} a_k^{(n)}\), then there exists a sequence without arbitrary long adjacent identical blocks such that no \(\lim_{k \to \infty} a_k^{(n)}\) exists. Let be \(\alpha \in (0, 1)\), then there exists such a sequence with \(\lim_{k \to \infty} a_k^{(1)} = \alpha\). Furthermore a class of sequences appearing in computer graphics is considered.

1. Introduction

In this section first the basic definitions are given, followed by a short survey of the remaining sections.

An alphabet \(\Sigma\) is a finite nonempty set, the elements of \(\Sigma\) are called symbols. \(\Sigma^*\) denotes the free monoid generated by \(\Sigma\). The elements of \(\Sigma^*\) are called words. The unit in \(\Sigma^*\) is denoted by \(\varepsilon\). The length of a word \(w \in \Sigma^*\) is denoted by \(|w|\) and is 0 if \(w = \varepsilon\) and \(n\) if \(w = a_1 \cdots a_n, a_i \in \Sigma\).

The mirror image of a word \(w \in \Sigma^*\) is denoted by \(w^R\) and is \(\varepsilon\) if \(w = \varepsilon\) and \(a_1 \cdots a_n a_n \cdots a_1\) if \(w = a_1 \cdots a_n, a_i \in \Sigma\).

An infinite sequence \(a_1a_2a_3 \cdots, a_i \in \Sigma\) is called \(\Sigma\)-sequence.

A substitution is a mapping \(\tau: \Sigma^* \to \mathcal{P}(\Sigma^*)\) such that the following conditions hold: \(\tau(\varepsilon) = \varepsilon\) and for each \(a \in \Sigma\) there exists \(L_a \subseteq \Sigma^*\), such that \(\tau(a_1 \cdots a_n) = L_{a_1} \cdots L_{a_n}\) for all \(a_1 \cdots a_n \in \Sigma^*\). Let \(\tau\) be a substitution such that for each \(a \in \Sigma\) \(\varepsilon \notin L_a\) holds. Then to each \(\Sigma\)-sequence \(\omega = a_1a_2a_3 \cdots\) corresponds the set \(\tau(\omega) = \{w_1w_2w_3 \cdots | w_i \in L_{a_i}\}\) of \(\Sigma_{2}\)-sequences.

Let \(\omega = a_1a_2a_3 \cdots\) be a \(\Sigma\)-sequence, \(a \in \Sigma, k \in \mathbb{N}\), then \(n_a(\omega)(k)\) denotes the number of symbols \(a\) in \(a_1 \cdots a_k\).

A word \(x \in \Sigma^*\) is called subword of a word \(w \in \Sigma^*\) (of a \(\Sigma\)-sequence \(\omega\)), if there are words \(y, z \in \Sigma^*\) (a word \(y \in \Sigma^*\) and a \(\Sigma\)-sequence \(\eta\)), such that \(w = yxz (\omega = y x n)\).

For \(\Sigma = \{0, 1\}\), \(\tau(0) = 1, \tau(1) = 0, \tau(w) (\tau(\omega))\) are abbreviated by \(\tilde{w}\) (\(\tilde{\omega}\)).

A \(\{0, 1\}\)-sequence \(\omega\) has arbitrary long adjacent identical blocks (is of unbounded repetition) provided that for all \(n \in \mathbb{N}\) there exists a subword \(ww\) of \(\omega\) where \(|w| = n\). A sequence not of this type is called sequence of bounded repetition.
The existence of sequences of unbounded repetition is evident. The second section contains a historical remark concerning the existence of sequences of bounded repetition; a special one is discussed in detail in Section 3, these examinations bring up some interesting arithmetical identities.

The relative frequencies of symbols 1 in sequences with bounded repetition are examined in Section 4.

Section 5 contains some results about operations on sequences of (un-) bounded repetition.

In the last section a class of sequences with unbounded repetition is related to a problem appearing in computer graphics.

2. Historical remark

It is well-known (Thue [12], Arshon [1], Hedlund and Morse [6]), that there are \{0, 1, 2\}-sequences containing no subword of the form \(ww\). Such \{0, 1, 2\}-sequences can be used in order to construct sequences of bounded repetition.

Entringer, Jackson and Schatz [4] have shown that there are \{0, 1\}-sequences having only subwords \(ww\) with \(|w| \leq 2\) and that this constant cannot be improved. The construction is based on a \{0, 1, 2\}-sequence containing no subword of the form \(ww\) and the substitution \(\tau(0) = 1010, \tau(1) = 1100, \tau(2) = 0111\).

It is remarked that the substitution \(\tau(0) = 0000, \tau(1) = 0101, \tau(2) = 1111\) is also possible.

A further sequence of bounded repetition can be constructed as in Section 3: The sequence 0000 \(\cdots\) is written down. Between every two symbols a gap is left. Now the sequence 1111 \(\cdots\) is filled in the gaps, where gaps of odd index are left free. In the remaining (infinitely many) gaps the sequence 000 \(\cdots\) is written, where again gaps of odd index are left free. This process (inserting 0's and 1's) is repeated ad infinitum. The \(n\)th element of this sequence can be obtained in the following way: if \(n = 2^{k+1}i + 2^k\), then \(a_n = k \pmod{2}\).

3. A special sequence with bounded repetition

Let be \(\omega = a_1 a_2 a_3 \cdots\), where \(a_n \in \{0, 1\}\). \(a_n = i \pmod{2}\) if \(n = 2^{k+1}i + 2^k\). Since each \(n \in \mathbb{N}\) can be uniquely written as \(n = 2^{k+1}i + 2^k\), \(\omega\) is well defined. (If the binary representation of \(n\) is \(w\sigma 10 \cdots 0, \sigma \in \{0, 1\}, \text{then } a_n = \sigma\). \(\omega\) can be defined as follows (see Jacobs and Keane [7]):

The sequence 0101 \(\cdots\) is written down, leaving a gap between every two symbols:

\[
\begin{align*}
a_1 & a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11} a_{12} a_{13} a_{14} \\
0 & 1 0 1 0 1 0 1 0
\end{align*}
\]
Infinite 0-1-sequences

Now the sequence 0101 \cdots is filled in the gaps, leaving free every second gap:

\[
\begin{array}{cccccccccccc}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}
\]

The remaining gaps are again filled by the sequence 0101 \cdots, leaving free every second gap:

\[
\begin{array}{cccccccccccc}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}
\]

This process is repeated ad infinitum.

**Theorem 3.1.** \(\omega\) is a sequence of bounded repetition.

**Proof.** It will be shown by induction on \(n \geq 6\), that \(\omega\) contains no word of the form \(xx\) with \(|x|=n\). (A separate discussion of the cases \(n = 6, 7, 8, 9\) and \(10\) is necessary.)

(i) \(n = 6\). In a set of \(6\) consecutive natural numbers there is always a \(k\) with \(k \equiv 1 \pmod{8}\) or \(k \equiv 5 \pmod{8}\). The binary representation of \(k\) ends in both cases with 01, therefore \(a_k = 0\). Thus the binary representation of \(k+6\) ends with 11, and so \(a_{k+6} = 1\). It follows that two consecutive words of length 6 in \(\omega\) differ at least at one position.

Similar arguments are used in the following cases:

(ii) \(n = 7\). In a set of \(7\) consecutive natural numbers there is always a \(k\) with \(k \equiv 3 \pmod{8}\) or \(k \equiv 6 \pmod{8}\). Therefore \(a_k = 1\), but \(a_{k+7} = 0\).

(iii) \(n = 8\). In a set of \(8\) consecutive natural numbers there is always a \(k\) with \(k \equiv 4 \pmod{16}\) or \(k \equiv 12 \pmod{16}\). In the first case \(a_k = 0\) and \(a_{k+8} = 1\), in the second case \(a_k = 1\) and \(a_{k+8} = 0\).

(iv) \(n = 9\). For \(k \equiv 5, 13 \pmod{16}\) \(a_k = 0\) and \(a_{k+9} = 1\).

(v) \(n = 10\). For \(k \equiv 4, 13 \pmod{16}\) \(a_k = 0\) and \(a_{k+10} = 1\).

(vi) Since \(a_2a_4a_6 \cdots = \omega = a_1a_2a_3 \cdots\), \(\omega\) contains to each subword \(xx\), where \(|x| = 2k\) already a subword yy, where \(|y| = k\) (yy is obtained by erasing all symbols of \(xx\) with odd index). Therefore the statement holds for even \(n\).

(vii) Let be \(n \geq 11\) an odd number and \(a_{i+1} \cdots a_{i+n}a_{i+n+1} \cdots a_{i+2n}\) a subword of \(\omega\) of the form \(xx\). Let \(k \in \{i+1, i+2\}\) be odd. Then

\[
a_k a_{k+1} \cdots a_{k+8} = \sigma a_{k+1} \sigma a_{k+3} \sigma a_{k+5} \sigma a_{k+7} \sigma = \alpha.
\]

Since \(k + n + 1\) is odd, in the same way it can be concluded that

\[
a_{k+n} a_{k+n+1} \cdots a_{k+n+8} = a_{k+n} \tau a_{k+n+2} \tau a_{k+n+4} \tau a_{k+n+6} \tau a_{k+n+8} = \beta
\]

and \(\alpha = \beta\) must hold. Therefore \(\alpha = \sigma \tau \sigma \tau \sigma \tau \sigma = \beta\). Without loss of generality let
be $\sigma = \tau$ (otherwise $\sigma$ is replaced by $\tau$ and $\tau$ by $\bar{\sigma}$). $a_{i+1} \cdots a_{i+n}$ contains a subword

$$\gamma = \sigma \sigma \sigma \sigma \sigma = a_{j+1} \cdots a_{j+8}.$$ 

Then there is a unique $r, j + 1 \leq r \leq j + 4$, such that $r = 2 \pmod{8}$ or $r = 6 \pmod{8}$. Like in (i)–(v) it can be concluded, that $a_r \neq a_{r+4}$, which is impossible because of the form of $\gamma$. (This reasoning also excludes, that $\omega$ contains a subword $xx$, where $|x| = 4$.)

In Theorem 3.3 it will be shown, that $\omega$ can be defined recursively (similar to Hedlund and Morse [6]).

The following lemma will then be used:

**Lemma 3.2.** $a_1 \cdots a_{2^n-1} = a_{2^n+1} \cdots a_{2^{n+1}-1}$ for all $n \in \mathbb{N}$.

**Proof.** Let be $1 \leq i \leq 2^n - 1$ and $w \sigma 10^k$ the binary representation of $i$. Then $1 \bar{w} \bar{\sigma} 10^k$ is the binary representation of $2^n + 1 - i$ and therefore $a_i = a_{2^n+1-i}$.

**Theorem 3.3.** Let be $\alpha_n, \beta_n, n \geq 1$ recursively defined as follows:

1. $\alpha_1 = 0, \quad \beta_1 = 1, \quad \alpha_{n+1} = \alpha_n 0 \beta_n, \quad \beta_{n+1} = \alpha_n 1 \beta_n, \quad n \geq 1.$

Then $a_1 \cdots a_{2^n-1} = \alpha_n$ for all $n \in \mathbb{N}$.

**Proof.** First, by induction on $n$, it is shown that $\alpha_n = \bar{\beta}_n^R$:

(i) $\alpha_1 = 0 = \bar{1}^R = \bar{\beta}_1^R$;

(ii) $\alpha_{n+1} = \alpha_n 0 \beta_n = \bar{\beta}_n 1, \quad \alpha_n = \alpha_n 1 \beta_n = \bar{\beta}_{n+1}^R$.

Now the statement of the theorem is proved by induction on $n$:

(i) $a_1 = 0 = \alpha_1$,

(ii) $a_1 \cdot \cdot a_{2^n-1} a_{2^n+1} \cdot \cdot a_{2^{n+1}-1} = a_1 \cdot \cdot a_{2^n-1} a_{2^n+1} \cdot \cdot a_{2^n-1}^R = \alpha_n 0 \alpha_n = \alpha_n 0 \beta_n = \alpha_{n+1}.$

In the rest of this section the numbers $n_1^{(\omega)}(k)$ and $\lim_{k \to \infty} n_1^{(\omega')}(k)/k$ are examined. (Since there is no danger of confusion, $n_1(k)$ will be written instead of $n_1^{(\omega)}(k)$.)

**Definition 3.4.** Let be $k \in \mathbb{N}_0$. The variation $v(k)$ of $k$ is defined recursively as follows: $v(0) = 0, \quad v(2j + i) = v(j) + \delta$, where $i, \delta \in \{0, 1\}$, $\delta = i + j \pmod{2}$.

Roughly spoken, $v(k)$ denotes the number of changes of consecutive digits in the binary representation of $k$, where the leftmost digit 1 counts as a change.

The following lemma shows a property of $v(k)$ which is used in the sequel.

**Lemma 3.5.** Let be $2^n \leq k < 2^{n+1}$. Then $v(k) = v(2^{n+1} - k - 1) + 1$. 
Infinite 0-1-sequences

Proof. By induction on $n$:
(i) If $n = 0$, then only $k = 1$ is possible and $v(1) = 1 = v(0) + 1$.
(ii) Let be $n \geq 1$ and $2^n \leq k < 2^{n+1}$. Then $k = 2j + 1$, $i \in \{0, 1\}$ and $2^{n-1} \leq j < 2^n$.

Let be $\delta, \delta' \in \{0, 1\}$, $\delta = i + j \pmod{2}$, $\delta' = 2^n - j - i \pmod{2}$. Then $\delta = \delta' \pmod{2}$
and

$v(k) = v(2j + i) = v(j) + \delta = v(2^n - j - 1) + 1 + \delta$

$= v(2^n - j - 1) + \delta' + 1 = v(2^n - j - 1) + 1 - i + 1$

$= v(2^{n+1} - 2j - i - 1) + 1 = v(2^{n+1} - k - 1) + 1$.

Now the numbers $n_i(k)$ and $v(k)$ can be related:

Theorem 3.6. $n_i(k) = \frac{1}{2}(k - v(k))$.

Proof. Let be $2^n \leq k < 2^{n+1}$. The statement is proved by induction on $n$:
(i) If $n = 0$, then $k = 1$ and $n_i(1) = 0 = \frac{1}{2}(1 - v(1))$.
(ii) Let be $n \geq 1$. The number of symbols 1 in $a_1 \cdots a_{2^n - 1}a_2 \cdots a_k$ can be determined in the following manner (it should be remembered that $a_{2^n} = 0$): In $a_1 \cdots a_{2^n - 1}$ occur exactly $n_i(2^n - 1) = \frac{1}{2}(2^n - 1 - v(2^n - 1))$ symbols 1. To this number the number $m$ of symbols 1 in $a_{2^n+1} \cdots a_{2^n+1}$ is added, and the number $m'$ of symbols 1 in $a_{k+1} \cdots a_{2^n+1}$ is subtracted. Lemma 3.2 implies that

$m = n_0(2^n - 1) = 2^n - 1 - n_1(2^n - 1)$

and

$m' = n_0(2^{n+1} - k - 1) = 2^{n+1} - k - 1 - n_1(2^{n+1} - k - 1)$.

Therefore

$n_i(k) = n_i(2^n - 1) + m - m'$

$= n_i(2^n - 1) + 2^n - 1 - n_1(2^n - 1) - 2^{n+1} + k + 1 + n_1(2^{n+1} - k - 1)$

$= -2^n + k + \frac{1}{2}(2^{n+1} - k - 1 - v(2^{n+1} - k - 1))$

$= \frac{1}{2}(k - 1 - (v(k) - 1))$

$= \frac{1}{2}(k - v(k))$.

Now it can be shown that the sequence $n_i(k)/k$ of the relative frequencies of symbols 1 in $\omega$ converges:

Theorem 3.7. $\lim_{k \to \infty} n_i(k)/k = \frac{1}{2}$.

Proof. Since $0 \leq v(k) \leq 1 + \lfloor k \rfloor$ always hold, it follows that

$\frac{1}{2}(k - 1 - \lfloor k \rfloor) \leq n_i(k) \leq \frac{1}{2} k$.
Therefore
\[
\frac{1}{2} \left( 1 - \frac{1 + \log k}{k} \right) \leq \frac{n_1(k)}{k} \leq \frac{1}{2}.
\]

Since \( \lim_{k \to \infty} (1 + \log k)/k = 0 \), the proof is finished.

Another way to compute \( n_1(k) \) shows

**Theorem 3.8.** \( n_1(k) = \sum_{i=0}^{\infty} \left\lfloor \frac{1}{2^i} \frac{k + 2^i}{2^{i+2}} \right\rfloor \).

**Proof.** First it should be noted that \( a_i = 1 \) if and only if there exists a number \( i \), such that \( s \equiv 3 \cdot 2^i \mod 2^i+2 \). (The binary representation of \( s \) must be of the form \( w110^i \).) Let \( i \) be fixed. Then there are \( \left\lfloor \frac{(k - 3 \cdot 2^i)/2^{i+2}}{2^{i+2}} \right\rfloor + 1 \) numbers \( s \), such that \( s \leq k \) and \( s \equiv 3 \cdot 2^i \mod 2^{i+2} \). (The following fact was used: For given \( n, r, m, 0 \leq r < m \), there are exactly \( \left\lfloor \frac{n - r}{m} \right\rfloor + 1 \) numbers \( t \), such that \( 0 \leq t \leq n \) and \( t \equiv r \mod m \).)

Furthermore
\[
\left\lfloor \frac{k - 3 \cdot 2^i}{2^{i+2}} \right\rfloor + 1 = \left\lfloor \frac{k - 3 \cdot 2^i + 2^{i+2}}{2^{i+2}} \right\rfloor = \left\lfloor \frac{k + 2^i}{2^{i+2}} \right\rfloor.
\]

A summation over \( i \) completes the proof.

Using Lemma 3.6 and Theorem 3.8 an interesting identity can be proved:

**Corollary 3.9.** \( \sum_{i=0}^{\infty} \left\lfloor \frac{(k + 2^i)/2^{i+2}}{2^{i+2}} \right\rfloor = \frac{1}{2} (k - v(k)) \).

In a similar way an other identity can be easily proved.

**Theorem 3.10.** \( \sum_{i=0}^{\infty} \left\lfloor \frac{(k + 2^i)/2^{i+1}}{2^{i+1}} \right\rfloor = k \).

**Proof.** Let \( \omega' = b_1 b_2 b_3 \cdots \) be defined as \( \omega \), but using \( 1111 \cdots \) instead of \( 0101 \cdots \). Then clearly \( n_1^{(\omega')(k)} = k \) holds for all \( k \).

\( n_1^{(\omega')(k)} \) can be determined as in the proof of Theorem 3.8: \( b_s = 1 \) if and only if there exists a number \( i \), such that \( s \equiv 2^i \mod 2^{i+1} \). (The binary representation of \( s \) must be of the form \( w10^i \).)

Let \( i \) be fixed. Then there are \( \left\lfloor \frac{(k - 2^i)/2^{i+1}}{2^{i+1}} \right\rfloor + 1 \) numbers \( s \) such that \( s \leq k \) and \( s \equiv 2^i \mod 2^{i+1} \). Since
\[
\left\lfloor \frac{k - 2^i}{2^{i+1}} \right\rfloor + 1 = \left\lfloor \frac{k + 2^i}{2^{i+1}} \right\rfloor,
\]
a summation over \( i \) completes the proof.
4. Some properties of sequences with (un-)bounded repetition

In the sequel it will be shown, that for each \( \alpha \in (0, 1) \) there exists a sequence with bounded repetition \( \omega = a_1 a_2 a_3 \cdots \), such that \( \lim_{k \to \infty} n_1^{(\omega)}(k)/k = \alpha \). To obtain this, it is necessary to make some preparations.

In the following \( \tau \) denotes the substitution \( \tau(0) = \{00, 01\} \), \( \tau(1) = \{11\} \).

**Lemma 4.1.** Let \( \omega \) be a sequence with bounded repetition. Then \( \tau(\omega) \) contains only sequences with bounded repetition.

**Proof.** Assume \( k \) to be a number such that \( \omega \) does not contain a subword \( w w \), where \( |w| \geq k \).

Assume that there is a sequence with unbounded repetition \( \eta \in \tau(\omega) \). Then \( \eta \) contains a subword \( a_{i+1} \cdots a_{i+m}a_{i+m+1} \cdots a_{i+2m} \) where \( m \geq 2k \).

It is necessary to distinguish the following cases:

(i) \( i \equiv 0 \pmod{2} \) and \( m \equiv 0 \pmod{2} \). Then there is a subword \( w w \) in \( \omega \), \( |w| \geq k \), corresponding by \( \tau \) to the subword \( a_{i+1} \cdots a_{i+2m} \).

(ii) \( i \equiv 0 \pmod{2} \) and \( m \equiv 1 \pmod{2} \). If \( a_{i+m}a_{i+m+1} = 0x \), then \( a_{i+2m} = 0 \) and therefore \( a_{i+2m-1} = 0 \), and therefore \( a_{i+m-1} = 0 \), etc. Because of this \( \omega \) contains a subword \( 0'0' \), where \( r \geq k \). If \( a_{i+m}a_{i+m+1} = 11 \), then \( a_{i+1} = 1 \), and therefore \( a_{i+2} = 1 \), and therefore \( a_{i+m+2} = 1 \), etc. Because of this \( \omega \) contains a subword \( 1'1' \), where \( r \geq k \).

(iii) \( i \equiv 1 \pmod{2} \) and \( m \equiv 0 \pmod{2} \). In this case \( \omega \) contains a subword of the form \( \sigma_1 w \sigma_2 w \sigma_3 \), where \( |w| \geq k - 1 \). From this it follows that \( \sigma_1 \neq \sigma_2 \) and \( \sigma_2 \neq \sigma_3 \) must hold. If \( \sigma_2 = 0 \), then \( \sigma_3 = 1 \) and therefore \( a_{i+m} = 0 \) and \( a_{i+2m} = 1 \); this is impossible. If \( \sigma_2 = 1 \), then \( \sigma_3 = 0 \) and therefore \( a_{i+m} = 1 \) and \( a_{i+2m} = 0 \); this is also impossible.

(iv) \( i \equiv 1 \pmod{2} \) and \( m \equiv 1 \pmod{2} \). In this case \( \omega \) contains a subword of the form \( \sigma_1 w \sigma_2 \sigma_3 w \sigma_4 \), where \( |w| \geq k-1 \). Therefore \( \sigma_1 \neq \sigma_3 \) or \( \sigma_2 \neq \sigma_4 \) must hold. Without loss of generality one can assume that \( \sigma_1 \neq \sigma_3 \). If \( \sigma_1 = 0 \), then \( \sigma_3 = 1 \) and therefore \( a_{i+m+1} = a_{i+m+2} = 1 \), etc. Because of this \( \omega \) contains a subword \( 0'0' \), where \( r \geq k \). The case \( \sigma_1 = 1, \sigma_3 = 0 \) can be discussed with similar arguments.

**Lemma 4.2.** Let \( \omega \) be a \( \{0, 1\} \)-sequence and \( \lim_{k \to \infty} n_1^{(\omega)}(k)/k = \alpha \). Then for each \( \beta \in [\alpha, \frac{1}{2}(\alpha + 1)] \) there is a \( \eta \in \tau(\omega) \), such that \( \lim_{k \to \infty} n_1^{(\eta)}(k)/k = \beta \).

**Proof.** If \( \beta = \alpha \), \( \eta \) is obtained from \( \omega \) by replacing each 0 by 00 and each 1 by 11. Now it is assumed, that \( \beta > \alpha \). Then it exists a \( k_0 \), such that \( n_1^{(\omega)}(k)/k \leq \beta \) holds for all \( k \geq k_0 \). All symbols 0 in \( \omega \) are replaced by 00 until \( k_0 \) is reached. Then all symbols 0 are replaced by 01, until a minimal \( k_1 \) is found, such that \( n_1^{(\eta)}(2k_1)/2k_1 \geq \beta \). (This is possible: if all but finitely many symbols in \( \omega \) are replaced by 01, then the sequence of relative frequencies of this new sequence...
converges to \( \frac{1}{2}(\alpha + 1) \). Beginning with index \( k_1 + 1 \) all symbols 0 are again replaced by 00, until a minimal \( k_2 \) is found, such that \( n_1^{(m)}(2k_2)/2k_2 \leq \beta \). This process is repeated. Clearly, the so constructed sequence \( \eta \) has the desired property. (Compare this construction with Knopp [8; p. 329].)

**Lemma 4.3.** Let \( \omega \) be a \( \{0, 1\} \)-sequence and \( \lim_{k \to \infty} n_1^{(\omega)}(k)/k = \alpha \). Then for each \( \beta \in [\alpha, 1) \) there exists a \( n \) and a \( \eta \in \tau^n(\omega) \), such that \( \lim_{k \to \infty} n_1^{(n)}(k)/k = \beta \).

**Proof.** Since the sequence \( (\alpha + 2^k - 1)/2^k \) increases strictly monotonously and converges to 1, there is a unique \( n \), such that

\[
\frac{\alpha + 2^n - 1}{2^n} \leq \beta < \frac{\alpha + 2^{n+1} - 1}{2^{n+1}} = \frac{(\alpha + 2^n - 1)/2^n + 1}{2}.
\]

Then there is a \( \eta' \in \tau^n(\omega) \), such that

\[
\lim_{k \to \infty} n_1^{(n')}(k)/k = \frac{\alpha + 2^n - 1}{2^n},
\]

Then, because of Lemma 4.2, there is a \( \eta \in \tau(\eta') \subseteq \tau^{n+1}(\omega) \), such that \( \lim_{k \to \infty} n_1^{(\eta)}(k)/k = \beta \).

**Theorem 4.4.** For each \( \beta \in [\frac{1}{2}, 1) \) there is a sequence with bounded repetition \( \eta \), such that \( \lim_{k \to \infty} n_1^{(\eta)}(k)/k = \beta \).

**Proof.** Let \( \omega \) be the sequence of Section 3. Then the statement is evident applying Lemma 4.3 to \( \omega \).

**Theorem 4.5.** For each \( \beta \in (0, 1) \) there is a sequence with bounded repetition \( \eta \), such that \( \lim_{k \to \infty} n_1^{(\eta)}(k)/k = \beta \).

**Proof.** The statement must be proved only for \( \beta \in (0, \frac{1}{2}] \). Let \( \omega \) be a sequence with bounded repetition and \( \lim_{k \to \infty} n_1^{(\omega)}(k)/k = 1 - \beta \). Then for \( \eta = \bar{\omega} \) the statement is true.

**Corollary 4.6. The set of sequences with bounded repetition has cardinality \( 2^{\aleph_0} \).**

This statement can be seen also in that way: From the work of Kakutani (cf. Gottschalk and Hedlund [5; p. 109]) there are \( 2^{\aleph_0} \) square-free \( \{0, 1, 2\} \)-sequences. This can be found also in Bean, Ehrenfeucht and McNulty [2]. Then a substitution as in Section 2 gives the result.

By \( \omega = a_1a_2a_3 \cdots \mapsto \Phi(\omega) = \sum a_j/2^j \), each \( \{0, 1\} \)-sequence can be associated with a real number in \([0, 1]\). Each real number which corresponds to a sequence with bounded repetition is non-normal in accordance to Borel [3]. (See Niven
Since the set of non-normal numbers is of measure 0, the following theorem holds:

**Theorem 4.7.** Let $M$ be the set of all sequences with bounded repetition. Then $\Phi(M)$ is of measure 0.

Finally, it is shown, that there are sequences with bounded repetition, for which the sequences of the "nth averages" do not converge.

**Definition 4.8.** Let $\omega = a_1a_2a_3\cdots$ be a $\{0, 1\}$-sequence and the sequences $a_1^{(n)}$, $a_2^{(n)}$, $a_3^{(n)}$, $\ldots$ of the nth average $(n \geq 0)$ defined by:

$$a_1^{(0)} = a_n, \quad a_1^{(n+1)} = \frac{1}{k} \sum_{k=1}^{n} a_k^{(n)}.$$

Then $a_1^{(k)} = n_1(k)/k$.

**Theorem 4.9.** There is a sequence with bounded repetition $\eta$, such that no

$$\lim_{k \to \infty} a_1^{(n)}$$

exists.

**Proof.** Let $\omega$ be the sequence of Section 3. $\eta$ will be constructed by applying the substitution $\tau$ to $\omega$ step by step. For this purpose let $\alpha, \beta$ be so that $\frac{1}{2} < \alpha < \beta < \frac{3}{4}$.

First step: Symbols 0 are replaced by 01 until $a_1^{(1)} \geq \beta$. Then symbols 0 are replaced by 00 until $a_1^{(1)} \leq \alpha$.

$k$th step: Symbols 0 are replaced by 01 until $a_1^{(1)} \geq \beta$, $a_2^{(2)} \geq \beta$ and $\ldots$ and $a_k^{(k)} \geq \beta$. Then symbols 0 are replaced by 00 until $a_1^{(1)} \leq \alpha$ and $\ldots$ and $a_k^{(k)} \leq \alpha$.

For each $k$ the sequence $a_1^{(k)}, a_2^{(k)}, a_3^{(k)}, \ldots$ contains infinitely many numbers $\leq \alpha$ and $\geq \beta$ and therefore it does not converge.

### 5. Operations on sequences with (un-) bounded repetition

The behaviour of sequences with (un-) bounded repetition is examined for the following operations: changes of finite character, mixing, addition mod 2.

**Lemma 5.1.** Let $\omega$ be a $\{0, 1\}$-sequence and $\sigma \in \{0, 1\}$. Then $\omega$ is a sequence with bounded repetition if and only if $\sigma \omega$ is a sequence with bounded repetition.

**Proof.** If $\sigma \omega$ is a sequence with bounded repetition, then there is a $k$, such that $\sigma \omega$ contains no subword $ww$, where $|w| \geq k$. Then $\omega$ does not contain such a subword and is therefore a sequence with bounded repetition.

Let conversely $\omega$ be a sequence with bounded repetition. Then there is a $k$, such that $\omega$ contains no subword $ww$, where $|w| \geq k$. Assuming $\sigma \omega$ to be a
sequence with unbounded repetition yields \( \sigma \omega = \alpha \alpha \eta_1 = \gamma \gamma \eta_2 \), where \( |\alpha| \equiv k \) and \( |\gamma| \equiv 2|\alpha| \). Then \( \gamma = \alpha \alpha \nu \), and \( \omega \) would contain \( \alpha \alpha \) as a subword.

**Theorem 5.2.** Let \( \omega \) be a \( \{0, 1\} \)-sequence and \( x, y \in \{0, 1\}^* \). Then \( x \omega \) is a sequence with bounded repetition if and only if \( y \omega \) is a sequence with bounded repetition.

**Proof.** It follows from Lemma 5.1 by induction, that \( x \omega \) is a sequence with bounded repetition if and only if \( \omega \) is a sequence with bounded repetition. By a similar argument it can be concluded, that \( \omega \) is a sequence with bounded repetition if and only if \( y \omega \) is a sequence with bounded repetition.

**Remark.** Theorem 5.2 shows, that by changes of finite character (deleting and inserting of finitely many symbols) of sequences with bounded repetition again sequences with bounded repetition are obtained.

Let \( \omega_2 \) be obtained from \( \omega_1 \) by changes of finite character, and \( k_i \) \((i = 1, 2)\) minimal, such that \( \omega_i \) contains no subword \( \omega \omega \), \( |\omega| \equiv k_i \). Then \( k_1 \) and \( k_2 \) can be quite different.

**Definition 5.3.** For \( \{0, 1\} \)-sequences \( \omega = a_1 a_2 a_3 \cdots \) and \( \eta = b_1 b_2 b_3 \cdots \) let

\[ \omega \square \eta = a_1 b_1 a_2 b_2 a_3 b_3 \cdots \]

**Theorem 5.4.** The sequences with unbounded repetition are not closed under \( \square \).

**Proof.** Let \( \omega \) be a sequence with bounded repetition, \( \tau_1(0) = 00 \), \( \tau_1(1) = 11 \), \( \tau_2(0) = 01 \), \( \tau_2(1) = 11 \). Then according to Lemma 4.1 \( \tau_1(\omega) = a_1 a_2 a_3 \cdots \) and \( \tau_2(\omega) = b_1 b_2 b_3 \cdots \) are sequences with bounded repetition. Let the sequences \( \eta_1 = a_1 a_2 a_3 \cdots \) and \( \eta_2 = b_1 b_2 b_3 \cdots \) be constructed as follows:

For all \( n \geq 0 \) let

\[
\begin{align*}
a_2^n \cdots a_{2^n+1-1} &= \begin{cases} 0^{2^n} & \text{if } n \text{ is even}, \\
a_2^n \cdots a_{2^n+1-1} & \text{if } n \text{ is odd}, \\
\end{cases} \\
b_2^n \cdots b_{2^n+1-1} &= \begin{cases} b_2^n \cdots b_{2^n+1-1} & \text{if } n \text{ is even}, \\
1^{2^n} & \text{if } n \text{ is odd}. \\
\end{cases}
\end{align*}
\]

Since \( \eta_1 \) and \( \eta_2 \) contain subwords \( 0^k 0^k \) and \( 1^k 1^k \) for infinitely many \( k \), they are sequences with unbounded repetition.

Now it is shown that \( \eta_1 \square \eta_2 \) is a sequence with bounded repetition. Assuming the contrary the following cases are possible:

(i) \( a_{r+1} \cdots a_{r+n} = a_{r+n+1} \cdots a_{r+2n} \).

Then \( a_{r+1} \cdots a_{r+n} = a_{r+n+1} \cdots a_{r+2n} \) and \( b_{r+1} \cdots b_{r+n} = b_{r+n+1} \cdots b_{r+2n} \), which is possible only for finitely many \( n \).
(ii) \[ b_{r+1} + a_{r+2} + \cdots + b_{r+n}a_{r+n+1} + b_{r+n+1}a_{r+n+2} + \cdots + b_{r+2n}a_{r+2n+1} \]
is discussed similar to (i).

(iii) \[ a_{r+1}b_{r+1} + \cdots + b_{r+n}a_{r+n+1} + b_{r+n+1}a_{r+n+2} + \cdots + a_{r+2n+1}b_{r+2n+1} \].

For a sufficiently large \( n \) there is a \( i \) \((1 \leq i \leq n)\), such that \( a_{r+i}a_{r+i+1} = 00 \) and therefore \( b_{r+i+n}b_{r+i+n+1} = 00 \), which is excluded by the construction of \( \eta_2 \).

(iv) \[ b_{r+1}a_{r+2} + \cdots + a_{r+n+1}b_{r+n+1} + a_{r+n+2}b_{r+n+2} + \cdots + b_{r+2n+1}a_{r+2n+2} \]
is discussed similar to (iii).

If \( \omega \) and \( \eta \) are sequences with bounded repetition, then it is quite possible, that \( \omega \Box \eta \) is a sequence with bounded repetition. (An example: \( \omega \Box \omega = \tau_1(\omega), \tau_1 \) from Theorem 5.4.) It could not be found out, whether or not this holds in general. However the following can be shown:

**Theorem 5.5.** For each sequence with bounded repetition \( \omega \) there is a \( \{0, 1\} \)-sequence \( \eta \), such that \( \omega \Box \eta \) is a sequence with unbounded repetition.

**Proof.** Let \( \omega = a_1a_2a_3 \cdots \) be a sequence with bounded repetition and \( \eta = b_1b_2b_3 \cdots \) be constructed as follows: for all \( n \geq 0 \) let be \( b_2^n \) be anyhow and

\[ b_{2^n+1} \cdots b_{2^n+1} = a_{2^n+2}a_{2^n+1} \cdots a_{2^n+1}a_{2^n+1} \cdots a_{2^n+2}. \]

Then \( \omega \Box \eta \) contains for all \( n \) the subword

\[ a_{2^n+1}a_{2^n+2}a_{2^n+1} \cdots a_{2^n+1}a_{2^n+2}a_{2^n+1}a_{2^n+2}a_{2^n+1} \cdots a_{2^n+1}a_{2^n+2}. \]

of the form \( ww \), the length of which is \( 2(2^n - 1) \).

Interpreting \( 0 \) and \( 1 \) as the elements of \( GF(2) \), and defining the addition of \( \{0, 1\} \)-sequences elementwise, it can be shown that neither the sequences of unbounded repetition nor the sequences with bounded repetition are closed under addition.

**Theorem 5.6.** There are sequences of bounded repetition \( \omega_1, \omega_2, \omega_3 \) and sequences with unbounded repetition \( \eta_1, \eta_2, \eta_3 \), such that

(i) \( \omega_1 + \omega_2 = \omega_3 \),

(iv) \( \omega_1 + \eta_2 = \eta_3 \),

(ii) \( \omega_1 + \omega_1 = \eta_1 \),

(v) \( \eta_2 + \eta_3 = \omega_1 \),

(iii) \( \omega_1 + \eta_1 = \omega_1 \),

(vi) \( \eta_1 + \eta_1 = \eta_1 \).

**Proof.** Let be \( \eta_1 = 0000 \cdots \). Then (ii), (iii) and (vi) are true. Let \( \omega \) be a sequence with bounded repetition, \( \omega_1 = \tau_2(\omega), \omega_2 = 1\omega_1 \) and \( \omega_3 = \tau_1(\omega) \) (\( \tau_i \) from Theorem 5.4). Since \( \tau_2(\omega) + 1\tau_2(\omega) = \tau_1(\omega) \), (i) holds.
Let be \( \omega_1 = a_1 a_2 a_3 \cdots \). Then \( \eta_2 = b_1 b_2 b_3 \cdots \) and \( \eta_3 = c_1 c_2 c_3 \cdots \) can be constructed in the following way:

\[
\begin{align*}
b_{2^i + \cdots + 2^i + 1 - 1} &= \begin{cases} 0^{2^i} & \text{if } n \text{ is even}, \\ d_{2^i + \cdots + 2^i + 1 - 1} & \text{if } n \text{ is odd}, \end{cases} \\
n_{2^i + \cdots + 2^i + 1 - 1} &= \begin{cases} 0^{2^i} & \text{if } n \text{ is odd}, \\ d_{2^i + \cdots + 2^i + 1 - 1} & \text{if } n \text{ is even}. \end{cases}
\end{align*}
\]

Then (iv) and (v) are true.

6. A class of sequences with unbounded repetition and a problem in computer graphics

Given a line \( y = ax, 0 \leq a \leq 1 \), which is to be drawn approximately for \( x \geq 0 \) by an \( s \)-directional-plotter in a way that the errors measured along the ordinate are minimal, the points \( (n, \lfloor an + \frac{1}{2} \rfloor) \) must be connected. The numbers \( b_n = \lfloor an + \frac{1}{2} \rfloor - \lfloor an \rfloor \) are in \( \{0, 1\} \) and correspond to the instructions for the plotter (in the \( n + 1 \)-th step a line between the points \( (n, m) \) and \( (n + 1, m + b_n) \) is drawn). (Cf. Prodinger et al. [11].)

The sequence \( b_1 b_2 b_3 \cdots \) is periodic with period \( q \), if and only if \( a = p/q \) is rational. For irrational \( a \), the sequence \( b_1 b_2 b_3 \cdots \) is not periodic, but the following theorem holds:

**Theorem 6.1.** For each \( a \in [0, 1] \) the sequence \( b_1 b_2 b_3 \cdots \) is a sequence with unbounded repetition.

**Proof.** If \( a \) is rational, the statement is evident. Let \( a \) be irrational. The sequence \( ak \mod 1 \) is dense in \( [0, 1) \) (Kuipers and Niederreiter [9; p. 23]).

Let be \( n' > 0 \). It will be shown that there is a \( n \geq n' \), such that the sequence \( b_1 b_2 b_3 \cdots \) contains a subword \( \mathbf{ww} \), where \( |w| = n \). Since \( ak \mod 1 \) is dense in \( [0, 1) \), there is a minimal \( n \geq n' \), such that

\[
0 < an \mod 1 < \lim_{1 \leq i \leq n} (ai \mod 1).
\]

Let be \( \varepsilon = an \mod 1 \),

\[
\begin{align*}
\delta_1 &= \min_{1 \leq i \leq n} \left( \frac{1}{2} - (ai \mod 1) \right), \\
\delta_2 &= \min_{1 \leq i \leq n} \left( ai \mod 1 - \frac{1}{2} \right).
\end{align*}
\]

Since \( n \) was chosen minimal, \( \varepsilon < \delta_1 + \delta_2 \) holds. Therefore there is a \( \delta \), where \( \varepsilon < \delta < \delta_1 + \delta_2 \) and \( \delta \geq \delta_1 \). Since \( 1 - \delta_2 < 1 - (\delta - \delta_1) \) and \( ak \) is dense, there exists a \( s \), such that \( as \mod 1 \in (1 - \delta_2, 1 - (\delta - \delta_1)) \).
Now it will be shown, that $b_{s+1} \cdots b_{s+n} = b_{s+n+1} \cdots b_{s+2n}$ holds: First, for $1 \leq i \leq n$,
\[
\alpha(s + n + 1) \mod{1} = (\alpha(s + i) + \varepsilon) \mod{1}
\]
holds. Since $(\alpha(s + i)) \mod{1} \notin \left[\frac{1}{2} - \varepsilon, \frac{1}{2}\right]$ it follows
\[
(\alpha(s + i) + \frac{1}{2}) \mod{1} \notin \left[1 - \varepsilon, 1\right].
\]
Let $i$ be chosen arbitrarily ($1 \leq i \leq n$). Let be
\[
h_1 = [\alpha(s + i - 1) + \frac{1}{2}] \quad \text{and} \quad h_2 = [\alpha(s + n + i - 1) + \frac{1}{2}].
\]
If $b_{s+i} = 1$, then
\[
\alpha(s + i) + \frac{1}{2} \in (n_1 + 1, n_1 + 2) \quad \text{and} \quad \alpha(s + n + i) + \frac{1}{2} \in (n_2 + 1, n_2 + 2).
\]
Therefore $b_{s+n+i} = 1$ holds. If $b_{s+i} = 0$, then $\alpha(s + i) + \frac{1}{2} \in (n_1, n_1 + 1 = \varepsilon)$ and therefore $\alpha(s + n + i) + \frac{1}{2} \in (n_2 + \varepsilon, n_2 + 1)$; from this $b_{s+n+i} = 0$.

**Remark.** In a very similar way one can show the following: Let $\gamma$ be an arbitrary real number, then the sequence $b_1', b_2', b_3', \ldots$ where $b_n' = [\alpha n + \gamma] = [\alpha(n = 1) + \gamma]$, is a sequence of unbounded repetition.

**Acknowledgement**

We wish to thank the referee for several helpful remarks, especially for calling our attention to Entringer, Jackson, Schatz [4].

**References**


