AN ELEMENTARY PROOF OF A SERIES EVALUATION IN TERMS OF HARMONIC NUMBERS

HELMUT PRODINGER

For positive integers \( j \), consider

\[
S(j) = \sum_{n \geq 1} \frac{1}{2^{2n-1}(2n-1)} \sum_{k} \binom{2n-1}{k} \frac{1}{k - j - n + \frac{1}{2}}.
\]

This quantity arose in [4] and was subsequently evaluated in [3]. Further proofs of the final formula (1) were given in [2, 1]. Here, we give an extremely short and simple proof.

For our analysis, it is better to replace the inner summation index \( k \) by \( 2n - 1 - k \), and consider

\[
T(j) = \sum_{n \geq 1} \frac{1}{2^{2n-1}(2n-1)} \sum_{k} \binom{2n-1}{k} \frac{1}{k + j - n + \frac{1}{2}},
\]

then \( S(j) = -T(j) \).

We start from the obvious formula

\[
\int_{0}^{1} t^{x-1}(1 + t)^{2n-1} dt = \sum_{k} \binom{2n-1}{k} \frac{1}{k + x},
\]

substitute \( x = j - n + \frac{1}{2} \) and sum:

\[
T(j) = \int_{0}^{1} t^{j-\frac{1}{2}}(1 + t)^{-1} \sum_{n \geq 1} \frac{1}{2^{2n-1}(2n-1)} t^{-n}(1 + t)^{2n} dt
\]

\[
= \int_{0}^{1} t^{j-\frac{1}{2}}(1 + t)^{-1} \frac{1 + t}{\sqrt{t}} \log \frac{1 + \sqrt{t}}{1 - \sqrt{t}} dt
\]

\[
= 2 \int_{0}^{1} w^{2j-1} \log \frac{1 + w}{1 - w} dw
\]

\[
= 4 \int_{0}^{1} w^{2j-1} \sum_{k \geq 1} \frac{w^{2k-1}}{2k - 1} dw
\]

\[
= 4 \sum_{k \geq 1} \frac{1}{(2k - 1)(2k + 2j - 1)}
\]

\[
= \frac{2}{j} \sum_{k \geq 1} \left( \frac{1}{2k - 1} - \frac{1}{2k + 2j - 1} \right)
\]

\[
= \frac{2}{j} \sum_{k=1}^{j} \frac{1}{2k - 1}.
\]

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Hence

\[ S(j) = -\frac{2}{j} \sum_{k=1}^{j} \frac{1}{2k-1}. \quad (1) \]

It is not hard to show that it is allowed to interchange integration and summation.

REFERENCES


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