Topic 11: Enumeration of graphs on surfaces

Scientific background The exact and asymptotic enumeration of graphs that are embedded or can be embedded in a given surface is a classical topic within enumerative combinatorics, starting with the influential work of Tutte in the 1960s [5, 6]. An important recent breakthrough was the asymptotic formula for the number of labelled planar graphs due to Giménez and Noy [4]. It is natural to consider graphs not only embedded in the plane, but also more general surfaces, and to study different families of graphs.

Hypotheses/Aims The general aim of enumeration of a given graph class $\mathcal{C}$ is to determine the number of elements of $\mathcal{C}$ with n vertices, either exactly or asymptotically as n tends to infinity. In this project, we aim to analyse various graph classes which are defined via embeddability on a given surface $\mathbb{S}$, such as graphs embeddable on $\mathbb{S}$ that satisfy degree conditions (see [2] for cubic graphs and multigraphs on an orientable surface) or directed graphs (e.g. using a toolkit developed in [1] for studying directed graphs with restrictions on their strong components).

Approaches/Methods The project combines topological aspects of graph theory with enumerative combinatorics. The characterisation of graphs that can be embedded in a given surface can often be translated to systems of functional equations for the corresponding generating functions. The first step will therefore be to obtain a suitable structural characterisation of the graph classes at hand.

The second part of our approach is to extract information from the aforementioned systems of equations. Here, analytic tools, most notably singularity analysis (see [3]), play a key role. Once the behaviour of the generating function at the complex singularities is understood, one can derive an asymptotic formula for the number of graphs of a certain size within a given family of graphs.

Time frame The methods for this topic originate from different fields—topological graph theory, graph structures, and enumerative combinatorics. We do not expect PhD candidates to have a background in all of these fields. If a candidate has excellent mathematical skills and a solid foundation in one of the fields mentioned above, then the knowledge of methods from other fields can be obtained during the initial part of the 4 year period.

Participating faculty members The project will be supervised by Philipp Sprüssel and Stephan Wagner.

Philipp Sprüssel is a Senior Lecturer (with habilitation) at the Institute of Discrete Mathematics at TU Graz. His main research focus is topological aspects of graphs and other combinatorial structures, both from an enumerative and probabilistic viewpoint.

Stephan Wagner has recently joined the Institute of Discrete Mathematics at TU Graz as a full professor. His research interests include combinatorial probability, enumerative and analytic combinatorics, and graph theory.

References

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