Asymptotics of the Moments of Extreme-value related distribution functions

Guy Louchard* Helmut Prodinger†

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Abstract

The asymptotic cost of many algorithms and combinatorial structures is related to the extreme-value Gumbel distribution $\exp(-\exp(-x))$. The following list is not exhaustive: Trie, Digital Search Tree, Leader Election, Adaptive Sampling, Counting Algorithms, trees related to the Register Function, Composition of Integers, some structures represented by Markov chains (Column-Convex Polyominoes, Carlitz Compositions), Runs and number of distinct values of some multiplicity in sequences of geometrically distributed random variables. Sometimes we can start from an exact (discrete) probability distribution, sometimes from an asymptotic analysis of the discrete objects (e.g., urn models) before establishing the relationship with the Gumbel distribution function. Also some Markov chains are either exactly and directly given by the structure itself, or as a limiting Markov process.

The main motivation of the paper is to compute the asymptotic distribution and the moments of the random variables in question. The moments are usually given by a dominant part and a small fluctuating part. We use Laplace and Mellin transforms and singularity analysis and aim for a unified treatment in all cases. Furthermore, our goal is a purely mechanical computation of dominant and fluctuating components, with the help of a computer algebra system. We provide each time the first three moments, but the treatment is (almost) completely automatic. We need some real analysis for the approximations and apart from that only easy complex analysis; simple poles and a few special functions.

1 Introduction.

The asymptotic cost of many algorithms is related to the extreme-value Gumbel distribution function (DF): $\exp(-\exp(-x))$. Also several combinatorial structures lead to the same behaviour.

A first aim of this paper is to come up with a classification of these objects. First of all, we can distinguish four classes depending on the *origin* of the objects. In the first class, the algorithms, we find the classical Trie, the Digital Search Tree, the Leader Election, the Adaptive Sampling, several Counting Algorithms (LogLog Counting, Approximate Counting, Probabilistic Counting). In the second class, the combinatorial structures, we have the trees related to the registers needed in arithmetic expressions evaluation, the composition of integers, the number of distinct values of some multiplicity in sequences of geometrically distributed random variables (RV). A third class is given by some structures represented by Markov chains: the column-convex polyominoes, the Carlitz compositions, the runs of geometrically distributed RV, the monotone runs of uniformly distributed integer RV. A last class is related to singularity analysis techniques leading also to a Gumbel distribution function. As an example, we can mention the $M/G/1$ queue.

A second way of classifying our objects is to separate the class where we start from an exact (discrete) probability distribution from the class where we first make an asymptotic analysis of the

*Université Libre de Bruxelles, Département d'Informatique, CP 212, Boulevard du Triomphe, B-1050 Bruxelles, Belgium, email: louchard@ulb.ac.be

†Mathematics Department, Stellenbosch University, 7602 Stellenbosch, South Africa, e-mail: hproding@sun.ac.za.
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discrete objects (for instance with urn models) before establishing the relationship with the Gumbel DF. Also some Markov chains are either exactly and directly given by the structure itself, or as a limiting Markov process.

The second and most important motivation of the paper is to compute the asymptotic distribution and the moments of our related RV. The moments are usually given by a dominant part and a small fluctuating part. There we use Laplace and Mellin transforms and singularity analysis. We wanted to obtain a unified treatment of all our cases.

Our aim is to derive a purely mechanical computation of dominant and fluctuating components, with the help of computer algebra system (we use Maple here). As an example, we provide each time the first three moments, (the third moment is very rarely computed in the literature) but the treatment is completely automatic (with some human guidance of course).

A last but not least motivation is to simplify the analytic treatment by using only easy complex analysis: only simple poles are needed, we do not use alternating series (so we do not need Rice's formula). A small number of analytic functions are the only tools we need. This should be compared with the complicated techniques sometimes used in previous papers (see, for instance, [1]).

The paper is organized as follows: in Section 2, we recall the definitions, notations and known properties related to the dominant part of the moments. Section 3 is devoted to a detailed analysis of the fluctuating part of the moments. Section 4 deals with the direct limit case, leading from the discrete distribution to the Gumbel DF or a convergent series of such. The crucial rate of convergence is given in each case (with classical analysis techniques). In Section 5, we compute the asymptotic form of the moments, in a mechanized way. Section 6 concludes the paper. Due to length constraints, some algorithms are detailed in the complete version of this paper [21]: Digital Search Tree, Leader Election, Approximate Counting, trees related to the Register Function. The number of distinct values of some multiplicity in sequences of geometrically distributed RV is analyzed in [22]. Also, the Markov chain approach (either in a limiting or an exact form) and the $M/G/1$ queue are considered in [21].

If we compare the approach in this paper with other ones that appeared previously, then we can notice the following. Traditionally, one would stay with exact enumerations as long as possible, and only at a late stage move to asymptotics. Doing this, one would, in terms of asymptotics, carry many unimportant contributions around, which makes the computations quite heavy, especially when it comes to higher moments. Here, however, approximations are carried out as early as possible, and this allows for streamlined (and often automatic) computations of the higher moments.

Let us finally point out some recent related works: in [11], S. Janson studies, by other methods, the characteristic function and moments of the integer-valued random variable $\lfloor X + \alpha \rfloor$, where X is a continuous random variable. In [28] and [24], P. Robert and H. Mohamed use Poisson processes to analyze some algorithms with an underlying tree structure, without resorting to complex analysis.

2 Definitions, notations and known properties.

Let us first give the notations we will use throughout the paper.

$$\begin{aligned}
L &:= \ln 2, \\
\log &:= \log_2, \\
\varepsilon &:= \text{ small real } > 0, \\
\tilde{\alpha} &:= \alpha/L, \\
Q(j) &:= \prod_{k=1}^{j}(1-2^{-k}), \quad Q(0)=1, \\
Q &:= Q(\infty) = 0.288788095087\ldots, \\
Q^{-}(j) &:= 1/Q(j), \\
R(j) &:= \frac{(-1)^{j+1}}{2^{j(j+1)/2}} Q^{-}(j), \ R(0) = -1.
\end{aligned}$$

These functions appear in many analyses of algorithms (see, for instance, Flajolet [3], Flajolet and Sedgewick [7], Louchard [19]).

The following facts will be frequently used:

$$\begin{aligned}
|Q - Q(i)| &= \mathcal{O}(2^{-i}),\\
Q(i) &\geq Q,\\
\ln(1-u) &> -u - u^2,\ u < 0.68,\\
(1-u)^n &< e^{-nu}, u \in]0,1[,\\
(1-u)^n &= e^{-nu}\left[1 - nu^2/2 + \mathcal{O}(nu^3)\right],\ u \in]0,1[,\\
(1-u)^n &= \left[1 - nu + n(n-1)u^2/2 + \mathcal{O}(nu^3)\right],\ u \in]0,1/n[,\\
\binom{n}{i}u^i(1-u)^{n-i} &= (nu)^i e^{-nu}/i![1 + \mathcal{O}(1/n) + \mathcal{O}(u) + \mathcal{O}(nu^2)], u \in]0,1[, i \text{ fixed},
\end{aligned}$$

this is the Poisson approximation.

For all integer-valued RV X_n analyzed in this paper, we set

$$p(j) := \Pr(X_n = j), \quad P(j) := \Pr(X_n \leq j).$$

Setting $\eta = j - \log n$, we will first compute f and F such that

$$p(j) \sim f(\eta), \quad P(j) \sim F(\eta), \quad n \to \infty,$$

and, of course,

$$f(\eta) = F(\eta) - F(\eta - 1).$$

Asymptotically, the distribution will be a periodic function of the fractional part of $\log n$. The distribution $P(j)$ does not converge in the weak sense, it does however converge along subsequences n_m for which the fractional part of $\log n_m$ is constant.

Next, we must show that

$$\mathbb{E}\left(X_n^k\right) = \sum_j j^k p(j) \sim \sum_j (\eta + \log n)^k [F(\eta) - F(\eta - 1)], \tag{1}$$

and/or

$$\mathbb{E}\left(\left[2^{X_n}/n\right]^d\right) = \sum_j (2^j/n)^d p(j) \sim \sum_j 2^{\eta d}[F(\eta) - F(\eta - 1)], \tag{2}$$

by computing a suitable rate of convergence. This is related to an uniform integrability condition (see Loève [17, Section 11.4].)

Finally we will use the following result from Hitczenko and Louchard [8] related to the dominant part of the moments (the '~' sign is related to the moments of the integer-valued RV X_n).

Lemma 2.1 *Let a (integer-valued) RV X_n be such that $\Pr(X_n - \log n \leq \eta) \sim F(\eta)$, where $F(\eta)$ is the DF of a continuous RV Z with mean m_1, second moment m_2, variance σ^2 and centered moments μ_k. Assume that $F(\eta)$ is either an extreme-value DF or a convergent series of such and that (1) is satisfied. Let*

$$\varphi(\alpha) = \mathbb{E}(e^{\alpha Z}) = 1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} m_k = e^{\alpha m_1}\lambda(\alpha),$$

say, with

$$\lambda(\alpha) = 1 + \frac{\alpha^2}{2}\sigma^2 + \sum_{k=3}^{\infty} \frac{\alpha^k}{k!}\mu_k.$$

Let w, κ's (with or without subscripts) denote periodic functions of $\log n$, with period 1 and with usually small mean and amplitude. Actually, these functions depend on the fractional part of $\log n$: $\{\log n\}$.

Then the corresponding moments of X_n are given by

$$\begin{aligned}
\mathbb{E}(X_n - \log n) &\sim \int_{-\infty}^{+\infty} x[F(x) - F(x-1)]dx + w_1 \\
&= \tilde{m}_1 + w_1, \quad \textit{with} \quad \tilde{m}_1 = m_1 + 1/2, \\
\textit{var}(X_n) &\sim \mathbb{E}(X_n - (\log n + \tilde{m}_1 + w_1))^2 \\
&\sim \int_{-\infty}^{+\infty} x^2[F(x) - F(x-1)]dx - \tilde{m}_1{}^2 + \kappa_2 \\
&= m_2 + m_1 + 1/3 - \tilde{m}_1{}^2 + \kappa_2 = \tilde{\sigma}^2 + \kappa_2, \quad \textit{with} \quad \tilde{\sigma}^2 = \sigma^2 + 1/12.
\end{aligned}$$

More generally, the centered moments of X_n are asymptotically given by $\tilde{\mu}_i + \kappa_i$, where

$$\Theta(\alpha) := 1 + \sum_{k=2}^{\infty} \frac{\alpha^k}{k!}\tilde{\mu}_k = \frac{2}{\alpha}\sinh(\alpha/2)\lambda(\alpha).$$

The neglected part is of order $1/n^\beta$ with $0 < \beta < 1$.

For instance, we derive

$$\begin{aligned}
\tilde{\mu}_2 &= \tilde{\sigma}^2 = \mu_2 + 1/12, \\
\tilde{\mu}_3 &= \mu_3, \\
\tilde{\mu}_4 &= \mu_4 + \sigma^2/2 + 1/80, \\
\tilde{\mu}_5 &= \mu_5 + 5/6\mu_3.
\end{aligned}$$

The moments of $X_n - \log n$ are asymptotically given by $\tilde{m}_i + w_i$, where the generating function of $\tilde{m}_i$ is given by

$$\phi(\alpha) := \int_{-\infty}^{\infty} e^{\alpha\eta} f(\eta)d\eta = 1 + \sum_{i=1}^{\infty} \frac{\alpha^i}{i!}\tilde{m}_i = \varphi(\alpha)\frac{e^\alpha - 1}{\alpha}. \tag{3}$$

The convergence domain for α will be studied in Section 5. This leads to

$$\begin{aligned}
\tilde{m}_1 &= m_1 + 1/2, \\
\tilde{m}_2 &= m_2 + m_1 + 1/3, \\
\tilde{m}_3 &= m_3 + 3m_2/2 + m_1 + 1/4;
\end{aligned}$$

w_i and κ_i will be analyzed in the next section. Note that

$$\Theta(\alpha) = \phi(\alpha)e^{-\alpha\tilde{m}_1}.$$

This leads to

$$\begin{aligned}
\tilde{\mu}_2 &= \tilde{m}_2 - \tilde{m}_1^2, \\
\tilde{\mu}_3 &= \tilde{m}_3 + 2\tilde{m}_1^3 - 3\tilde{m}_2\tilde{m}_1.
\end{aligned} \tag{4}$$

Also

$$\varphi(\alpha) = \int_{-\infty}^{+\infty} e^{\alpha\eta}F'(\eta)d\eta = -\alpha\int_{-\infty}^{+\infty} e^{\alpha\eta}F(\eta)d\eta,$$

with suitable boundary conditions (which are satisfied in all examples we present).

3 The fluctuating components.

3.1 The moments of $X_n - \log n$.

To analyze the periodic component w_i to be added to the moments $\tilde{m}_i$ we return to the proof of the Lemma 2.1 given in Hitczenko and Louchard [8]. For instance, under (1),

$$\mathbb{E}(X_n - \log n) \sim E^{(1)}(n) = \sum_{j=1}^{\infty}[F(j - \log n) - F(j - \log n - 1)][j - \log n]. \tag{5}$$

Set $y = 2^{-x}$ and $G(y) = F(x)$. Equation (5) becomes a *harmonic sum*

$$E^{(1)}(n) := \sum_j [G(n/2^j) - G(n/2^{j+1})][-\log(n/2^j)],$$

the Mellin transform of which is (for a good reference on Mellin transforms, see Flajolet et al. [5] or Szpankowski [29])

$$\frac{2^s}{1-2^s}\Upsilon_1^*(s), \tag{6}$$

and

$$\begin{aligned}\Upsilon_1^*(s) &= \int_0^{\infty} y^{s-1}[G(y) - G(y/2)][-\log y]dy \\ &= \int 2^{-sx}[F(x) - F(x-1)]xLdx.\end{aligned}$$

Then

$$\Upsilon_1^*(s) = L\ \phi'(\alpha)\big|_{\alpha=-Ls}\,. \tag{7}$$

The fundamental strip of (6) is usually of the form $s \in \langle -C_1, 0\rangle$, $C_1 > 0$. This will be detailed in Section 5. Set also

$$\Upsilon_0^*(s) = L\ \phi(\alpha)|_{\alpha=-Ls}\,,\ \Upsilon_0^*(0) = L.$$

We assume now that all poles of $\frac{2^s}{1-2^s}\Upsilon_1^*(s)$ are simple, which will be the case in all our examples, and given by $s = 0$, $s = \chi_l$, with $\chi_l := 2l\pi i/L$, and, in some cases (DST for example), $s = C_3$, $C_3 > 0$, but we will check that the residue of $\frac{2^s}{1-2^s}\Upsilon_1^*(s)$ at C_3 is 0.

Using

$$E^{(1)}(n) = \frac{1}{2\pi i}\int_{C_2-i\infty}^{C_2+i\infty} \frac{2^s}{1-2^s}\Upsilon_1^*(s)n^{-s}ds, \quad -C_1 < C_2 < 0,$$

the asymptotic expression of $E^{(1)}(n)$ is obtained by moving the line of integration to the right, for instance to the line $\Re = C_4 > 0$, taking residues into account. This gives

$$E^{(1)}(n) = -\operatorname{Res}\Big[\frac{2^s}{1-2^s}\Upsilon_1^*(s)n^{-s}\Big]\Big|_{s=0} - \sum_{l\neq 0}\operatorname{Res}\Big[\frac{2^s}{1-2^s}\Upsilon_1^*(s)n^{-s}\Big]\Big|_{s=\chi_l} + \mathcal{O}(n^{-C_4}), \tag{8}$$

under a "slow increase property" (see below).

The residue at $s = 0$ gives of course

$$\tilde{m}_1 = \Upsilon_1^*(0)/L = \phi'(0).$$

The other residues lead to

$$w_1 = \sum_{l\neq 0}\Upsilon_1^*(\chi_l)e^{-2l\pi i\log n}/L. \tag{9}$$

More generally,

$$\mathbb{E}(X_n - \log n)^k \sim \tilde{m}_k + w_k,$$

with

$$w_k = \sum_{l \neq 0} \Upsilon_k^*(\chi_l) e^{-2l\pi i \log n}/L, \tag{10}$$

and

$$\Upsilon_k^*(s) = L\, \phi^{(k)}(\alpha)\Big|_{\alpha=-Ls}.$$

The residues analysis is similar to the previous one.

To compute the periodic component κ_i to be added to the centered moments $\tilde{\mu}_i$, we first set

$$\mathfrak{m}_1 := \tilde{m}_1 + w_1.$$

The variance of $X_n - \log n$ is asymptotically given by

$$\mathbb{E}[(X_n - \log n - \mathfrak{m}_1)^2] \sim \tilde{m}_2 + w_2 - \mathfrak{m}_1^2 = \tilde{\mu}_2 + \kappa_2.$$

The third centered moment is asymptotically given by

$$\mathbb{E}[(X_n - \log n - \mathfrak{m}_1)^3] \sim \tilde{m}_3 + w_3 - 3(\tilde{m}_2 + w_2)\mathfrak{m}_1 + 3\mathfrak{m}_1^3 - \mathfrak{m}_1^3 = \tilde{\mu}_3 + \kappa_3.$$

More generally, we start from

$$\phi(\alpha) := 1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \tilde{m}_k = \varphi(\alpha) \frac{e^\alpha - 1}{\alpha}.$$

We replace $\tilde{m}_k$ by $\tilde{m}_k + w_k$, leading to

$$\phi_p(\alpha) = \phi(\alpha) + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} w_k.$$

But it is easy to check that

$$\sum_{l \neq 0} \phi(-L\chi_l) e^{-2l\pi i \log n} = 0,$$

so we obtain

$$\begin{aligned} \phi_p(\alpha) &= \phi(\alpha) + \sum_{k=0}^{\infty} \sum_{l \neq 0} \phi^{(k)}(\alpha)\Big|_{\alpha=-L\chi_l} e^{-2l\pi i \log n} \frac{\alpha^k}{k!} \\ &= \phi(\alpha) + \sum_{l \neq 0} \phi(\alpha - L\chi_l) e^{-2l\pi i \log n} \\ &= \sum_{l \neq 0} \phi(\alpha - L\chi_l) e^{-2l\pi i \log n}. \end{aligned} \tag{11}$$

Finally, we compute

$$\Theta_p(\alpha) = \phi_p(\alpha) e^{-\alpha \mathfrak{m}_1} = 1 + \sum_{k=2}^{\infty} \frac{\alpha^k}{k!} (\tilde{\mu}_k + \kappa_k) = \Theta(\alpha) + \sum_{k=2}^{\infty} \frac{\alpha^k}{k!} \kappa_k, \tag{12}$$

leading to the (exponential) generating function (GF) of κ_k. This leads to

$$\begin{aligned} \kappa_2 &= w_2 - w_1^2 - 2\tilde{m}_1 w_1, \\ \kappa_3 &= 6\tilde{m}_1^2 w_1 + 6\tilde{m}_1 w_1^2 + 2w_1^3 - 3\tilde{m}_2 w_1 - 3\tilde{m}_1 w_2 - 3w_1 w_2 + w_3. \end{aligned} \tag{13}$$

All algebraic manipulations of this paper will be mechanically performed by Maple. We will give explicit expressions for $\tilde{\mu}_3$ and κ_3.

It will appear that $\Upsilon_k^*(s)$ are analytic functions (in some domain), depending on classical functions such as Γ, ζ, $\psi(k,s)$ (the $(k+1)$-gamma function) and an entire function $N(s)$ which is the analytic continuation of

$$\sum_{j\geq 1}(-1)^{\nu(j)}/j^s,$$

where $\nu(j)$ denotes the number of ones in the binary representation of j. The justification of (8) is classically done by considering the rectangle (for details, see Section 5.1)

$$[C_2-(k+1/2)\chi_l,\ C_2+(k+1/2)\chi_l,\ C_4+(k+1/2)\chi_l,\ C_4-(k+1/2)\chi_l].$$

But we know that $\Gamma(s)$ decreases exponentially in the direction $i\infty$:

$$|\Gamma(\sigma+it)|\sim\sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}.$$

Also, we have a "slow increase property" for all functions we encounter: let $s=\sigma+it$,

$$|\zeta(s)|=\mathcal{O}(|t|^{1-\sigma}),\ \sigma<1,$$

and a uniform version exists,

$$N(s)=\mathcal{O}(1+|s|^4),\ \sigma>-1.$$

(See Flajolet and Martin [6]).

It is not always evident that the limiting function $F(\eta)$ is indeed a distribution function. But here we can use the following lemma from Janson [11]. Assume that Y_n is a sequence of integer-valued random variables that is monotone. If, for every sequence of integers k_n,

$$\Pr(Y_n\leq k_n)=F(k_n-a_n)+o(1), n\to\infty,$$

for a right-continuous function F with $\lim_{x\to-\infty}F(x)=0$, $\lim_{x\to\infty}F(x)=1$ and a sequence a_n such that $a_n\to\infty$ and $a_{n+1}-a_n\to 0$ as $n\to\infty$, then F is a distribution function.

These conditions will be satisfied in our examples.

3.2 The moments of $2^{X_n}/n$.

Let us analyze the moments $\mathbb{E}[(2^{X_n}/n)^d]$. This amounts to computing

$$\sum_j[G(n/2^j)-G(n/2^{j+1})](2^j/n)^d,$$

the Mellin transform of which is

$$\frac{2^s}{1-2^s}\Upsilon_{1,d}^*(s), \tag{14}$$

with (we put an extra 1 index in the expressions of this Section to make the difference with the moments of $X_n-\log n$)

$$\begin{aligned}\Upsilon_{1,d}^*(s) &= \int_0^\infty y^{s-1}[G(y)-G(y/2)]y^{-d}dy\\ &= \int 2^{-sx}2^{dx}[F(x)-F(x-1)]Ldx\\ &= L\ \phi(\alpha)|_{\alpha=L(d-s)}\,.\end{aligned}$$

So (14) leads to

$$\mathbb{E}[(2^{X_n}/n)^d]\sim\tilde{m}_{1,d}+w_{1,d}, \tag{15}$$

with

$$\begin{aligned}\tilde{m}_{1,d} &= \phi(Ld),\\ w_{1,d} &= \sum_{l\neq 0}\Upsilon_{1,d}^*(\chi_l)e^{-2l\pi i\log n}/L.\end{aligned} \tag{16}$$

Similarly, we derive

$$\begin{aligned}
\phi_1(\alpha) &= 1+\sum_{k=1}^{\infty}\frac{\alpha^k}{k!}\tilde{m}_{1,k} = 1+\sum_{k=1}^{\infty}\frac{\alpha^k}{k!}\phi(Lk),\\
\phi_{p1}(\alpha) &= 1+\sum_{k=1}^{\infty}\frac{\alpha^k}{k!}[\tilde{m}_{1,k}+w_{1,k}]\\
&= 1+\sum_{k=1}^{\infty}\frac{\alpha^k}{k!}\left[\phi(Lk)+\sum_{l\neq 0}\phi(\alpha)|_{\alpha=L(k-\chi_l)}\,e^{-2l\pi i\log n}\right]\\
&= \sum_{0}^{\infty}\frac{\alpha^k}{k!}\left[\phi(Lk)+\sum_{l\neq 0}\phi(\alpha)|_{\alpha=L(k-\chi_l)}\,e^{-2l\pi i\log n}\right],\\
\mathfrak{m}_{1,1} &= \tilde{m}_{1,1}+w_{1,1} = \phi(L)+\sum_{l\neq 0}\phi(\alpha)|_{\alpha=L(1-\chi_l)}\,e^{-2l\pi i\log n},\\
\Theta_1(\alpha) &= \phi_1(\alpha)e^{-\alpha\tilde{m}_{1,1}} = 1+\sum_{k=2}^{\infty}\frac{\alpha^k}{k!}\tilde{\mu}_{1,k}, \qquad (17)\\
\Theta_{p1}(\alpha) &= \phi_{p1}(\alpha)e^{-\alpha\mathfrak{m}_{1,1}} = \Theta_1(\alpha)+\sum_{k=2}^{\infty}\frac{\alpha^k}{k!}\kappa_{1,k}. \qquad (18)
\end{aligned}$$

4 Direct limits. The rate of convergence.

We will not give details on the description of algorithms, in order to keep the paper's length within reasonable limits. All algorithms are completely explained in the references we provide. We deal here with the direct limit case, leading from the discrete distribution to the Gumbel DF or a convergent series of such. We will compute $f(\eta)$ and $F(\eta)$. The crucial rate of convergence is given in each case (with classical analysis techniques).

4.1 The depth of the Trie.

From Pittel [25], Jacquet and Régnier [10], Louchard [18], we have

$$P(j) = (1-1/2^j)^n. \qquad (19)$$

Let $\eta = j-\log n$, this leads to

$$F(\eta) = \exp(-2^{-\eta}). \qquad (20)$$

First we must bound $|P(j)-F(\eta)|$. Let $1/2<\beta<1$. We will consider three ranges

- for $j<\beta\log n$, $P(j)$ is small. Indeed
$$P(j)\leq \exp(-n^{1-\beta}).$$
- for $\beta\log n\leq j<2\log n$, we have
$$|P(j)-F(\eta)| = F(\eta)\mathcal{O}(n/n^{2\beta}) = \mathcal{O}(1/n^{2\beta-1}).$$
- for $j=2\log n+x$, $x\geq 0$, we have
$$|P(j)-F(\eta)| = \mathcal{O}(n/(n^4 2^{2x})) = \mathcal{O}(2^{-2x}/n^3).$$

To justify (1), we must bound the difference between the moments of X_n and the moments based on $F(\eta)$:

$$\begin{aligned}
&\Big|\sum_j j^k\big[[P(j)-P(j-1)]-[F(\eta)-F(\eta-1)]\big]\Big| \\
&\leq \quad \Big[\mathcal{O}((\beta\log n)^{k+1}\exp(-n^{1-\beta})) \\
&\qquad + (2\log n)^{k+1}\mathcal{O}(1/n^{2\beta-1}) + \mathcal{O}\Big(\sum_{x\geq 0}(2\log n+x)^k 2^{-2x}/n^3\Big)\Big] = \mathcal{O}(1/n^{2\beta-1-\varepsilon}).
\end{aligned}$$

4.2 The Adaptive Sampling algorithm.

R denotes the last (left) bucket size, with value r. From Flajolet [4] and Louchard [20], we have

$$\begin{aligned}
p(r,j) &= \binom{n}{r}\Big(\frac{1}{2^j}\Big)^r\Big(1-\frac{1}{2^j}\Big)^{n-r}\Big[1-\sum_{k=0}^{b-r}\binom{n-r}{k}\Big(\frac{1}{2^j}\Big)^k\Big(1-\frac{1}{2^j}\Big)^{n-r-k}\Big], \\
p(j) &= \sum_{r=0}^{b} p(r,j).
\end{aligned}$$

We are interested in the random variables: $W := X_n - \log n$ and $Z := r2^{X_n}/n$. So we set $\eta = j - \log n$, this leads to

$$f(r,\eta) = \exp(-2^{-\eta})\frac{2^{-r\eta}}{r!}\Big[1-\exp(-2^{-\eta})\sum_{k=0}^{b-r}\frac{2^{-k\eta}}{k!}\Big].$$

Also we derive

$$P(r,j) = \binom{n}{r}\sum_{u=1}^{j}\Big(\frac{1}{2^u}\Big)^r\Big(1-\frac{1}{2^u}\Big)^{n-r}\Big[1-\sum_{k=0}^{b-r}\binom{n-i}{k}\Big(\frac{1}{2^u}\Big)^k\Big(1-\frac{1}{2^u}\Big)^{n-r-k}\Big]$$

and

$$F(r,\eta) = \sum_{l=0}^{\infty}\exp(-2^{-(\eta-l)})\frac{2^{-r(\eta-l)}}{r!}\Big[1-\exp(-2^{-(\eta-l)})\sum_{k=0}^{b-r}\frac{2^{-k(\eta-l)}}{k!}\Big]. \tag{21}$$

First we must bound $|p(j)-f(\eta)|$. Of course, we can treat each index k, r separately. We will consider the first term in $p(r,j)$, denoted by $p_1(r,j)$. The other terms are analyzed similarly.

To analyze the moments of X_n, we must bound $|\sum_j j^k[p_1(r,j)-f_1(r,\eta)]|$. This is done like in the Trie case (details are given in [21]). To analyze the moments of Z, i.e., $\mathbb{E}\left[(r2^{X_n}/n)^d\right]$, we must bound

$$S_1 := \sum_j (2^j/n)^d |p_1(r,j)-f_1(r,\eta)|.$$

Again, details are given in [21]. Actually, it is shown in Louchard [20], due to some cancellations, that we must only have $d \leq b$.

4.3 The LogLog Counting algorithm.

This is one of the most delicate cases. From Durand and Flajolet [2] we must analyze the moments $\mathbb{E}(\Delta^d)$ of $\Delta := m/n2^J$, where $J := \sum_1^m J_i$ and J_i has distribution (19) with n now given by ν_i : the number of keys in bucket i. ν_i is BIN$[\tilde{n}, \tilde{n}(1-1/m))]$ with $\tilde{n} := n/m$ and m is a design parameter. Actually, $\{\nu_1 \ldots \nu_m\}$ is given by a multinomial distribution. We know that we can construct a "box"

$$[\tilde{n}-\tilde{n}^\theta, \tilde{n}+\tilde{n}^\theta]^m, 1/2 < \theta < 1, \tag{22}$$

such that, by large deviation analysis, the probability that $\{\nu_1 \dots \nu_m\}$ is outside this box is bounded by $\exp(-C\tilde{n}^{2\theta-1})$. We rewrite Δ as

$$\Delta = 2^{\sum X_i/m}, \text{ with } X_i := J_i - \log_2(\tilde{n}).$$

Let us first assume that ν_i is exactly given by its mean $\tilde{n}$. Dropping the i-index from X_i and the $\tilde{}$ sign from $\tilde{n}$ for convenience, we first compute $\mathbb{E}(2^{\tau X}) = \mathbb{E}((2^J/n)^\tau)$. We proceed as in the Adaptive Sampling case. We must bound

$$S_1 := \sum_j (2^j/n)^\tau |P(j) - F(\eta)|,$$

where $P(j)$, $F(\eta)$ are given by (19), (20). Actually, we will see that we must have $\tau < 1$. Details are given in [21]. Now we assume that we are inside the box (22) and we must bound

$$S_2 := \sum_j (2^j/n)^\tau \left| P(j) - (1 - 1/2^j)^{n+\xi n^\theta} \right|,$$

with $|\xi| < 1$. Again, details are given in [21] Now we consider the case $\nu > n + n^\theta$. We can show that

$$S_3 := \sum_j p(j)(2^j/n)^\tau \exp(-Cn^{2\theta-1}), \text{ with } p(j) \text{ based on } \nu,$$

is small. Details are given in [21] Now we consider the case $0 < \nu < n - n^\theta$. We have here $0 < \nu/n < 1$. The analysis proceeds like in the previous cases.

As $\Delta = 2^{\sum X_i/m}$, we see that we must now have $d < m$.

4.4 Product forms: Probabilistic Counting and first empty part in Compositions of integers.

The Probabilistic Counting is analyzed in Flajolet and Martin [6] (compare also [15]). Compositions of integers are analyzed in Hitczenko and Louchard [8], where it is shown that the first empty part value is asymptotically distributed as the first empty urn in an urn model where we throw n GEOM(1/2) RV. This is exactly equivalent to the Probabilistic Counting. Moreover, it is shown in [8] that the urns are asymptotically independent with probability $(1 - 1/2^i)^n$ for the urn i to be empty. So, setting (we omit n in $A_\cdot(j)$ for convenience)

$$A_1(j) := \prod_{i=1}^{j} [1 - (1 - 1/2^i)^n] \le 1,$$

we obtain here

$$P(j) = 1 - A_1(j). \tag{23}$$

In Flajolet and Martin [6], they use

$$P^*(j) = 1 - \sum_{i=0}^{2^j} (-1)^{\nu(i)} (1 - i/2^j)^n,$$

where again $\nu(j)$ denotes the number of ones in the binary representation of j.

$P(j)$ and $P^*(j)$ are asymptotically equivalent (see Section 5.5 for details), but we will start here from (23) to illustrate the manipulation of product forms. We have

$$\begin{aligned}
\eta &= j - \log n, \\
A_2(j) &:= \prod_{k=0}^{j-1} \left[1 - \exp(-e^{-L(\eta-k)})\right] \leq 1, \\
p(j) &= (1-1/2^j)^n A_1(j-1), \\
A_3(\eta) &:= \prod_{k=0}^{\infty} [1 - \exp(-e^{-L(\eta-k)})] \leq 1, \\
F(\eta) &:= 1 - A_3(\eta), \\
f(\eta) &:= \exp(-e^{-L\eta}) A_3(\eta - 1).
\end{aligned} \tag{24}$$

4.4.1

Let us first bound $\Delta_1 := |A_1(j) - A_2(j)|$. We have

$$A_1(j) = \prod_{k=0}^{j-1} \left[1 - \exp\left(-e^{-L(\eta-k)}\right) \left[1 - \frac{n}{2^{1+2(j-k)}} + \mathcal{O}\left(\frac{n}{2^{3(j-k)}}\right)\right]\right].$$

Set

$$\varphi(x) := \exp(-e^{-Lx}).$$

We derive

$$\begin{aligned}
A_1(j) &= \prod_{k=0}^{j-1} \left[[1 - \varphi(\eta-k)] \left[1 + \frac{\varphi(\eta-k)}{2(1-\varphi(\eta-k))} \left[e^{-2L(\eta-k)}/n + \mathcal{O}(e^{-3L(\eta-k)}/n^2)\right]\right]\right] \\
&= A_2(j) A_4(j),
\end{aligned}$$

with

$$A_4(j) := \prod_{k=0}^{j-1} \left[1 + \frac{\varphi(\eta-k)}{2(1-\varphi(\eta-k))} \left[e^{-2L(\eta-k)}/n + \mathcal{O}(e^{-3L(\eta-k)}/n^2)\right]\right].$$

We have the following equivalents:

$$\begin{aligned}
\eta - k < 0 \quad &: \quad \frac{\varphi(\eta-k)}{1-\varphi(\eta-k)} e^{-2L(\eta-k)} < \frac{\varphi(\eta-k)}{\varphi(0)} e^{-2L(\eta-k)}, \\
\eta - k \geq 0 \quad &: \quad \varphi(\eta-k) = 1 - e^{-L(\eta-k)} + \mathcal{O}(e^{-2L(\eta-k)}), \\
&\qquad 1 - \varphi(\eta-k) = e^{-L(\eta-k)} + \mathcal{O}(e^{-2L(\eta-k)}), \\
&\qquad \frac{\varphi(\eta-k)}{1-\varphi(\eta-k)} e^{-2L(\eta-k)} = \varphi(\eta-k) \left[e^{-L(\eta-k)} + \mathcal{O}(e^{-2L(\eta-k)})\right].
\end{aligned}$$

All these functions are bounded. Now

$$\begin{aligned}
\ln(A_4(j)) &= \mathcal{O}\left[\sum_{k=0}^{j-1} \Big[\varphi(\eta-k)e^{-L(\eta-k)}/n + \varphi(\eta-k)e^{-2L(\eta-k)}/n \right. \\
&\qquad \left. + \ \varphi^2(\eta-k)e^{-2L(\eta-k)}/n^2 + \varphi^2(\eta-k)e^{-4L(\eta-k)}/n^2\Big]\right] = \mathcal{O}(1/n), \text{ uniformly on } \eta.
\end{aligned}$$

So

$$A_4(j) = 1 + \mathcal{O}(1/n), \text{ uniformly on } \eta,$$

and

$$\Delta_1 = A_2(j)\mathcal{O}(1/n).$$

4.4.2

Let us now bound $\Delta_2 := |A_2(j) - A_3(\eta)|$. We have

$$A_3(\eta) - A_2(j) = A_2(j)[1 - A_5(j)],$$

with

$$A_5(j) := \prod_{k=j}^{\infty}[1 - \exp(-e^{-L(\eta-k)})].$$

We obtain, independently of j,

$$A_5 := \prod_{l=0}^{\infty}[1 - \exp(-ne^{Ll})].$$

But, as n is large,

$$\ln(A_5) \geq -\sum_{0}^{\infty}\exp(-ne^{Ll}) + \mathcal{O}\Big(\sum_{0}^{\infty}\exp(-2ne^{Ll})\Big) \sim -\int_0^{\infty} e^{-nu}\frac{du}{Lu} \sim -\frac{e^{-n}}{Ln}.$$

So

$$A_5 \geq 1 - \frac{e^{-n}}{Ln},$$

and

$$\Delta_2 \leq A_2(j)\frac{e^{-n}}{Ln},$$

uniformly on j.

4.4.3

Now we bound $\Delta_3 := |P(j) - F(\eta)|$, $\eta \geq 0$. We have

$$\begin{aligned}\Delta_3 &= A_2(j)\mathcal{O}(1/n) = \left(1 - \exp(e^{-L\eta})\right)\mathcal{O}(1/n) \\ &= \left[1 - \left(1 - e^{-L\eta} + \mathcal{O}(e^{-2L\eta})\right)\right]\mathcal{O}(1/n) = e^{-L\eta}\left[1 + \mathcal{O}(e^{-L\eta})\right]\mathcal{O}(1/n).\end{aligned}$$

4.4.4

Now we can proceed as in Section 4.3. We will analyze $\mathbb{E}((2^{X_n}/n)^\tau)$ (the analysis of the RV X_n is similar). Let us first assume that ν_i is exactly given by its mean $\tilde{n}$. We must bound

$$S_{1,1} := \sum_j (2^j/n)^\tau |p(j) - f(\eta)|,$$
$$S_{1,2} := \sum_j (2^j/n)^\tau |P(j) - F(\eta)|.$$

Actually, it can be shown that we must have $\tau < 1$. Now we assume that we are inside the box (22) and we must bound

$$S_{1,1} := \sum_j (2^j/n)^\tau |p_1(j) - p_2(j)|,$$
$$S_{1,2} := \sum_j (2^j/n)^\tau |P_1(j) - P_2(j)|,$$

where $p_1(j), P_1(j)$ are based on n and $p_2(j), P_2(j)$ are based on $n + \xi n^\theta, |\xi| < 1$. Now we consider the case $\nu > n + n^\theta$. We can show that

$$S_3 := \sum_j p(j)(2^j/n)^\tau \exp(-Cn^{2\theta-1}), \text{ with } p(j) \text{ based on } \nu,$$

is small. Details of all this are given in [21].

5 Direct limits. The moments.

We compute here the asymptotic form of the moments, in a mechanized way. Let us first recall the notations we use:

$$\begin{aligned}\varphi(\alpha) &= \int_{-\infty}^{+\infty} e^{\alpha\eta}F'(\eta)d\eta = e^{\alpha m_1}\lambda(\alpha),\\ m_1 &= \int_{-\infty}^{+\infty} \eta F'(\eta)d\eta,\\ \lambda(\alpha) &= 1+\frac{\alpha^2}{2}\sigma^2+\sum_{k=3}^{\infty}\frac{\alpha^k}{k!}\mu_k.\end{aligned}$$

In particular, the Gumbel DF: $F(\eta)=\exp(-e^{-\eta})$ leads to

$$\begin{aligned}\varphi(\alpha) &= \Gamma(1-\alpha), \Re(\alpha)<1,\\ m_1 &= \gamma,\\ m_2 &= \pi^2/6+\gamma^2,\\ m_3 &= 2\zeta(3)+\pi^2\gamma/2+\gamma^3,\\ \phi(\alpha) &= \varphi(\alpha)(e^{\alpha}-1)/\alpha \text{ , there is no singularity at } \alpha=0,\\ \tilde{m}_1 &= \gamma+1/2,\\ \tilde{m}_2 &= \gamma+1/3+\pi^2/6+\gamma^2,\\ \tilde{m}_3 &= 1/4+2\zeta(3)+\pi^2\gamma/2+\gamma^3+\gamma+\pi^2/4+3\gamma^2/2,\\ \lambda(\alpha) &= e^{-\alpha m_1}\varphi(\alpha),\\ \sigma^2 &= \pi^2/6,\\ \mu_3 &= 2\zeta(3),\\ \mu_4 &= 3\pi^4/20,\\ \Theta(\alpha) &= \frac{2}{\alpha}\sinh(\alpha/2)\lambda(\alpha),\\ \tilde{\mu}_2 &= \pi^2/6+1/12,\\ \tilde{\mu}_3 &= 2\zeta(3).\end{aligned}$$

The fundamental strip for

$$\frac{2^s}{1-2^s}\Upsilon_k^*(s)$$

is $\Re(s)\in\langle -1/L, 0\rangle$.

For further use, we also need

$$\int_{-\infty}^{+\infty} e^{\alpha\eta}e^{\eta}[\exp(-e^{-\eta})-1]d\eta = \frac{\Gamma(1-\alpha)}{\alpha(1+\alpha)}.$$

5.1 The depth of the Trie.

We have, from (20),

$$\begin{aligned}F(\eta) &= \exp(-e^{-L\eta}),\\ \varphi(\alpha) &= \Gamma(1-\tilde{\alpha}), \text{ with } \tilde{\alpha}:=\alpha/L,\ \Re(\alpha)<L.\end{aligned}$$

This leads to

$$\begin{aligned}
m_1 &= \gamma/L,\\
m_2 &= (\pi^2/6+\gamma^2)/L^2,\\
m_3 &= (2\zeta(3)+\pi^2\gamma/2+\gamma^3)/L^3,\\
\tilde{m}_1 &= \gamma/L+1/2,\\
\tilde{m}_2 &= \gamma/L+1/3+(\pi^2/6+\gamma^2)/L^2,\\
\tilde{m}_3 &= 1/4+(2\zeta(3)+\pi^2\gamma/2+\gamma^3)/L^3+\gamma/L+(\pi^2/2+3\gamma^2)/(2L^2),\\
\sigma^2 &= \pi^2/(6L^2),\\
\mu_3 &= 2\zeta(3)/L^3,\\
\mu_4 &= 3\pi^4/(20L^4),\\
\tilde{\mu}_2 &= \pi^2/(6L^2)+1/12,\\
\tilde{\mu}_3 &= 2\zeta(3)/L^3.
\end{aligned}$$

Let us now turn to the fluctuating components. The fundamental strip for (6) is $\Re(s)\in\langle -1,0\rangle$. First of all, (7) and (9) lead to

$$w_1 = -\sum_{l\neq 0}\Gamma(\chi_l)e^{-2l\pi i\log n}/L. \tag{25}$$

Equations (11) and (12) lead, after the usual simplifications necessary to help Maple, to

$$\kappa_2 = -2\gamma w_1/L - w_1^2 + 2\sum_{l\neq 0}\Gamma(\chi_l)\psi(\chi_l)e^{-2l\pi i\log n}/L^2,$$

where ψ is the digamma function. $\tilde{m}_1$, $\tilde{\mu}_2$, w_1 and κ_2 are identical to the expressions given in Knuth [16], Kirschenhofer and Prodinger [13], (again after some simplifications). Next we obtain

$$\begin{aligned}
\kappa_3 &= -6(w_1+\gamma/L)\sum_{l\neq 0}\Gamma(\chi_l)\psi(\chi_l)e^{-2l\pi i\log n}/L^2 - 3\sum_{l\neq 0}\Gamma(\chi_l)\psi^2(\chi_l)e^{-2l\pi i\log n}/L^3\\
&\quad -3\sum_{l\neq 0}\Gamma(\chi_l)\psi(1,\chi_l)e^{-2l\pi i\log n}/L^3 + 2w_1^3 + (3\gamma^2-\pi^2/2)w_1/L^2 + 6\gamma w_1^2/L,
\end{aligned}$$

where $\psi(1,x)$ is the trigamma function.

Fluctuations in the higher moments.

The ubiquitous fluctuating functions are usually written as a Fourier series with mean zero (coefficient with index 0 missing). However, looking at w_1^2, w_1^3, ..., these functions do not have mean zero, even though w_1 has. In the following, we demonstrate how the mean can be extracted and the Fourier coefficients can be explicitly expressed. We show this for the second power, where the computations had appeared already elsewhere, but then deal with the third power; these computations are new (and quite involved).

The following approach ("residue calculus") to evaluate $[\delta^2]_0$ (which is the constant term[1] in the Fourier expansion of the square of the periodic function $\delta(x)$) seems to be the most appropriate one. See the survey [26]. We start with the following example:

$$\delta(x) = \frac{1}{L}\sum_{k\neq 0}\Gamma(-\chi_k)e^{2\pi ikx} = -w_1(x).$$

Find a function $F(z)$ so that $[\delta^2]_0$ is (apart from a few extra terms) the sum of the residues along the imaginary axis. Here, take

$$F(z) = \frac{L}{2^z-1}\Gamma(-z)\Gamma(z).$$

[1] We will use notations like $[\delta^2]_k$ to indicate the kth Fourier coefficient.

For convenience, we define

$$\int_{(x)} := \int_{x-i\infty}^{x+i\infty}.$$

If we set

$$I_1 = \frac{1}{2\pi i}\int_{(1/2)} F(z)dz,$$

then by shifting and collecting residues,

$$I_1 = \frac{1}{2\pi i}\int_{(-1/2)} F(z)dz + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

What happens here is often called *closing the box*, see e. g. [29, 23]. One integrates along a rectangle with corners $\pm\frac{1}{2} \pm iM$. One can evaluate it by collecting the residues inside the rectangle. And one can let the parameter M go to infinity. In this type of problems, the integrals along the horizontal lines disappear, and we can express one integral along a vertical line by an integral along another vertical line, plus a few residues.

Now one writes[2]

$$\frac{1}{e^z-1} = -1 - \frac{1}{e^{-z}-1}$$

and gets, by a simple change of variable $z := -z$,

$$I_1 = -\frac{L}{2\pi i}\int_{(-1/2)} \Gamma(-z)\Gamma(z)dz - I_1 + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

The integral

$$I_2 = -\frac{L}{2\pi i}\int_{(-1/2)} \Gamma(-z)\Gamma(z)dz$$

can be computed by collecting the residues left to the line $\Re z = -\frac{1}{2}$, viz.

$$I_2 = -\frac{L}{2\pi i}\int_{(-1/2)} \Gamma(-z)\Gamma(z)dz = -L\sum_{l\geq 1}\frac{(-1)^l}{l!}(l-1)! = L^2.$$

Altogether we have

$$2I_1 = L^2 + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

On the other hand, integral I_1 is also the sum of the negative residues right of the line $\Re z = \frac{1}{2}$, i. e.,

$$I_1 = -L\sum_{l\geq 1}\frac{(-1)^l}{l!(2^l-1)}(l-1)! = -L\sum_{l\geq 1}\frac{(-1)^l}{l(2^l-1)}.$$

Combining these results, we get

$$-2L\sum_{l\geq 1}\frac{(-1)^l}{l(2^l-1)} = L^2 + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

This is the identity we wanted.

[2] As S. Janson has pointed out, it is not (always) necessary to take the term '-1' out, as the resulting series (over residues on the left) would still (though not absolutely) converge. However, over the years, we encountered similar problems, where one would get series like $1-2+3-4+\cdots$, so it is just "safer" to proceed as we decided to do here; compare the survey [26].

With not much more effort one can also compute the coefficients $[\delta^2]_k$, for $k \neq 0$. For this, one works with the function

$$F(z) = \frac{L}{2^z - 1}\Gamma(-z - \chi_k)\Gamma(z).$$

One obtains

$$\begin{aligned} a_k := [\delta^2]_k &= \frac{1}{L^2}\sum_{j\neq 0,\,\neq k}\Gamma(-\chi_j)\Gamma(-\chi_k + \chi_j) \\ &= \frac{2}{L}\sum_{l\geq 1}\frac{(-1)^l\Gamma(-\chi_k + l)}{l!(2^l - 1)} + \frac{2}{L^2}\Gamma(-\chi_k)\big(\psi(-\chi_k) + \gamma\big). \end{aligned}$$

Now we want the (Fourier-)coefficients of the third power of $\delta(x)$. We start with

$$\begin{aligned} [\delta^3(x)]_0 &= \frac{1}{L}\sum_{k\neq 0} a_k\Gamma(\chi_k) \qquad (26) \\ &= \frac{2}{L^3}\sum_{k\neq 0}\Gamma(-\chi_k)\big(\psi(-\chi_k) + \gamma\big)\Gamma(\chi_k) + \frac{2}{L^2}\sum_{l\geq 1}\frac{(-1)^l}{l!(2^l - 1)}\sum_{k\neq 0}\Gamma(-\chi_k + l)\Gamma(\chi_k). \end{aligned}$$

Let us first consider

$$\Xi := \sum_{k\neq 0}\Gamma(-\chi_k)\big(\psi(-\chi_k) + \gamma\big)\Gamma(\chi_k).$$

Define, as usually, the function

$$F(z) = \frac{L}{2^z - 1}\Gamma(-z)\big(\psi(-z) + \gamma\big)\Gamma(z)$$

and set

$$I := \frac{1}{2\pi i}\int_{(1/2)}\frac{L}{2^z - 1}\Gamma(-z)\big(\psi(-z) + \gamma\big)\Gamma(z)dz.$$

Then

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{(-1/2)}\frac{L}{2^z - 1}\Gamma(-z)\big(\psi(-z) + \gamma\big)\Gamma(z)dz \\ &\quad + \sum_{k\neq 0}\Gamma(-\chi_k)\big(\psi(-\chi_k) + \gamma\big)\Gamma(\chi_k) + \zeta(3) \\ &= \Xi + \zeta(3) + \frac{L}{2\pi i}\int_{(-1/2)}\left[-1 - \frac{1}{2^{-z} - 1}\right]\Gamma(-z)\big(\psi(-z) + \gamma\big)\Gamma(z)dz \\ &= \Xi + \zeta(3) - \frac{L}{2\pi i}\int_{(-1/2)}\Gamma(-z)\big(\psi(-z) + \gamma\big)\Gamma(z)dz \\ &\quad - \frac{L}{2\pi i}\int_{(-1/2)}\frac{1}{2^{-z} - 1}\Gamma(-z)\big(\psi(-z) + \gamma\big)\Gamma(z)dz \\ &= \Xi + \zeta(3) - L\sum_{l\geq 1}\frac{(-1)^l H_{l-1}}{l} - L\sum_{l\geq 1}\frac{(-1)^l H_{l-1}}{(2^l - 1)l} \\ &= \frac{1}{2\pi i}\int_{(1/2)}\frac{L}{2^z - 1}\Gamma(-z)\big(\psi(-z) + \gamma\big)\Gamma(z)dz \\ &= -L\sum_{l\geq 1}\frac{1}{2^l - 1}\frac{(-1)^{l-1}H_{l-1}}{l} + L^2\sum_{l\geq 1}\frac{2^l}{(2^l - 1)^2}\frac{(-1)^{l-1}}{l}. \end{aligned}$$

From this we find

$$\Xi = -\zeta(3) + \frac{L^3}{2} - 2L\sum_{l\geq 1}\frac{(-1)^{l-1}H_{l-1}}{l(2^l-1)} + L^2\sum_{l\geq 1}\frac{(-1)^{l-1}2^l}{l(2^l-1)^2}.$$

(Note that, since this is a tiny quantity, we find an approximation of $\zeta(3)$ in this way.)

For our application in (26), it must come with a factor $2/L^3$:

$$1 - \frac{2\zeta(3)}{L^3} - \frac{4}{L^2}\sum_{l\geq 1}\frac{(-1)^{l-1}H_{l-1}}{l(2^l-1)} + \frac{2}{L}\sum_{l\geq 1}\frac{(-1)^{l-1}2^l}{l(2^l-1)^2}.$$

Now let us concentrate on the other term in (26). For $l \geq 1$, look at

$$\Xi = \sum_{k\neq 0}\Gamma(-\chi_k + l)\Gamma(\chi_k).$$

Take

$$F = \frac{L}{2^z-1}\Gamma(l-z)\Gamma(z)$$

and

$$I = \frac{1}{2\pi i}\int_{(1/2)}\frac{L}{2^z-1}\Gamma(l-z)\Gamma(z)dz.$$

Then the usual computation gives

$$\begin{aligned}
I &= \Xi - (l-1)!H_{l-1} - (l-1)!\frac{L}{2} \\
&\quad - \frac{L}{2\pi i}\int_{(-1/2)}\Gamma(l-z)\Gamma(z)dz - \frac{L}{2\pi i}\int_{(-1/2)}\frac{1}{2^{-z}-1}\Gamma(l-z)\Gamma(z)dz \\
&= \Xi - (l-1)!H_{l-1} - (l-1)!\frac{L}{2} \\
&\quad - L\sum_{j\geq l}(-1)^{j-l}\frac{(j-1)!}{(j-l)!} - L\sum_{j\geq 1}\frac{1}{2^j-1}\frac{(l+j-1)!(-1)^j}{j!} \\
&= \Xi - (l-1)!H_{l-1} - (l-1)!\frac{L}{2} \\
&\quad - L(l-1)!\sum_{j\geq 0}(-1)^j\binom{j+l-1}{j} - L(l-1)!\sum_{j\geq 1}\frac{(-1)^j}{2^j-1}\binom{l+j-1}{j} \\
&= \Xi - (l-1)!H_{l-1} - (l-1)!\frac{L}{2} \\
&\quad - L(l-1)!2^{-l} - L(l-1)!\sum_{j\geq 1}\frac{(-1)^j}{2^j-1}\binom{l+j-1}{j}.
\end{aligned}$$

But also

$$I = L\sum_{j\geq l}\frac{1}{2^j-1}(-1)^{j-l}\frac{(j-1)!}{(j-l)!} = L(l-1)!\sum_{j\geq l}\frac{1}{2^j-1}(-1)^{j-l}\binom{j-1}{l-1},$$

and so

$$\begin{aligned}
\Xi &= (l-1)!H_{l-1} + (l-1)!\frac{L}{2} + L(l-1)!2^{-l} \\
&\quad + L(l-1)!\sum_{j\geq 1}\frac{(-1)^j}{2^j-1}\binom{l+j-1}{j} + L(l-1)!\sum_{j\geq l}\frac{1}{2^j-1}(-1)^{j-l}\binom{j-1}{l-1}.
\end{aligned}$$

For our application, it comes with a factor $\frac{2}{L^2}\sum_{l\ge1}\frac{(-1)^l}{l!(2^l-1)}$:

$$\begin{aligned}
&\frac{2}{L^2}\sum_{l\ge1}\frac{(-1)^l H_{l-1}}{l(2^l-1)}+\frac{1}{L}\sum_{l\ge1}\frac{(-1)^l}{l(2^l-1)}+\frac{2}{L}\sum_{l\ge1}\frac{(-1)^l}{l(2^l-1)}2^{-l}\\
&+\frac{2}{L}\sum_{l\ge1}\frac{(-1)^l}{l(2^l-1)}\sum_{j\ge1}\frac{(-1)^j}{2^j-1}\binom{l+j-1}{j}\\
&+\frac{2}{L}\sum_{l\ge1}\frac{(-1)^l}{l(2^l-1)}\sum_{j\ge l}\frac{1}{2^j-1}(-1)^{j-l}\binom{j-1}{l-1}.
\end{aligned}$$

The grand summing up:

$$\begin{aligned}
[\delta^3(x)]_0 &= 1-\frac{2\zeta(3)}{L^3}-\frac{4}{L^2}\sum_{l\ge1}\frac{(-1)^{l-1}H_{l-1}}{l(2^l-1)}+\frac{2}{L}\sum_{l\ge1}\frac{(-1)^{l-1}2^l}{l(2^l-1)^2}\\
&\quad+\frac{2}{L^2}\sum_{l\ge1}\frac{(-1)^l H_{l-1}}{l(2^l-1)}+\frac{1}{L}\sum_{l\ge1}\frac{(-1)^l}{l(2^l-1)}+\frac{2}{L}\sum_{l\ge1}\frac{(-1)^l}{l(2^l-1)}2^{-l}\\
&\quad+\frac{2}{L}\sum_{l\ge1}\frac{(-1)^l}{l(2^l-1)}\sum_{j\ge1}\frac{(-1)^j}{2^j-1}\binom{l+j-1}{j}\\
&\quad+\frac{2}{L}\sum_{l\ge1}\frac{(-1)^l}{l(2^l-1)}\sum_{j\ge l}\frac{1}{2^j-1}(-1)^{j-l}\binom{j-1}{l-1}.
\end{aligned}$$

After several simplifications this becomes

$$\begin{aligned}
[\delta^3(x)]_0 &= -1-\frac{2\zeta(3)}{L^3}-\frac{1}{L}\sum_{l\ge1}\frac{(-1)^l}{l(2^l-1)}+\frac{6}{L^2}\sum_{l\ge1}\frac{(-1)^l H_{l-1}}{l(2^l-1)}+\frac{2\ln 3}{L}\\
&\quad+\frac{2}{L}\sum_{l,j\ge1}\frac{(-1)^{l+j}}{(l+j)(2^l-1)}\left[\frac{1}{2^j-1}+\frac{1}{2^{j+l}-1}\right]\binom{l+j}{j}.
\end{aligned}$$

This has been checked numerically and gives the tiny value 9.428177×10^{-25}.

Now attack $[\delta^3(x)]_j$, for $j\neq0$.

$$\begin{aligned}
[\delta^3(x)]_j &= \frac{1}{L}\sum_{k\neq j}a_k\Gamma(\chi_{k-j})\\
&= \frac{1}{L}\sum_{k\neq j,\neq0}a_k\Gamma(\chi_{k-j})+\frac{1}{L}a_0\Gamma(-\chi_j)\\
&= \frac{1}{L}\sum_{k\neq j,\neq0}\left[\frac{2}{L^2}\Gamma(-\chi_k)\big(\psi(-\chi_k)+\gamma\big)+\frac{2}{L}\sum_{l\ge1}\frac{(-1)^l\Gamma(-\chi_k+l)}{l!(2^l-1)}\right]\Gamma(\chi_{k-j})\\
&\quad+\frac{1}{L}\left[\frac{\pi^2}{6L^2}+\frac{1}{12}-1-\frac{2}{L}\sum_{l\ge1}\frac{(-1)^{l-1}}{l(2^l-1)}\right]\Gamma(-\chi_j)\\
&= \frac{2}{L^3}\sum_{k\neq j,\neq0}\Gamma(-\chi_k)\big(\psi(-\chi_k)+\gamma\big)\Gamma(\chi_{k-j})\\
&\quad+\frac{2}{L^2}\sum_{k\neq j,\neq0}\sum_{l\ge1}\frac{(-1)^l\Gamma(-\chi_k+l)}{l!(2^l-1)}\Gamma(\chi_{k-j})\\
&\quad+\left[\frac{\pi^2}{6L^3}+\frac{1}{12L}-\frac{1}{L}-\frac{2}{L^2}\sum_{l\ge1}\frac{(-1)^{l-1}}{l(2^l-1)}\right]\Gamma(-\chi_j).
\end{aligned}$$

The third sum stands as it is, so let us start with the first one,

$$\Xi := \sum_{k\neq j,\neq 0} \Gamma(-\chi_k)\big(\psi(-\chi_k)+\gamma\big)\Gamma(\chi_{k-j})$$

Consider

$$F(z) = \frac{L}{2^z-1}\Gamma(-z)\big(\psi(-z)+\gamma\big)\Gamma(z-\chi_j)$$

and set

$$I := \frac{1}{2\pi i}\int_{(1/2)} \frac{L}{2^z-1}\Gamma(-z)\big(\psi(-z)+\gamma\big)\Gamma(z-\chi_j)dz.$$

Then

$$\begin{aligned}
I &= \frac{1}{2\pi i}\int_{(-1/2)} \frac{L}{2^z-1}\Gamma(-z)\big(\psi(-z)+\gamma\big)\Gamma(z-\chi_j)dz + \Xi \\
&\quad - \frac{1}{12}\Big(18\Psi(1,-\chi_j) + 18\big(\psi(-\chi_j)+\gamma\big)^2 + L^2 - \pi^2\Big)\Gamma(-\chi_j) \\
&= -\frac{1}{2\pi i}\int_{(-1/2)} \frac{L}{2^{-z}-1}\Gamma(-z)\big(\psi(-z)+\gamma\big)\Gamma(z-\chi_j)dz \\
&\quad - \frac{L}{2\pi i}\int_{(-1/2)} \Gamma(-z)\big(\psi(-z)+\gamma\big)\Gamma(z-\chi_j)dz \\
&\quad + \Xi - \frac{1}{12}\Big(18\Psi(1,-\chi_j) + 18\big(\psi(-\chi_j)+\gamma\big)^2 + L^2 - \pi^2\Big)\Gamma(-\chi_j) \\
&= -\frac{1}{2\pi i}\int_{(-1/2)} \frac{L}{2^{-z}-1}\Gamma(-z-\chi_j)\big(\psi(-z-\chi_j)+\gamma\big)\Gamma(z)dz \\
&\quad - \frac{L}{2\pi i}\int_{(-1/2)} \Gamma(-z)\big(\psi(-z)+\gamma\big)\Gamma(z-\chi_j)dz \\
&\quad + \Xi - \frac{1}{12}\Big(18\Psi(1,-\chi_j) + 18\big(\psi(-\chi_j)+\gamma\big)^2 + L^2 - \pi^2\Big)\Gamma(-\chi_j) \\
&= -L\sum_{l\geq 1} \frac{(-1)^l}{l!(2^l-1)}\Gamma(l-\chi_j)\big(\psi(l-\chi_j)+\gamma\big) \\
&\quad - L\sum_{l\geq 0} \frac{(-1)^l}{l!}\Gamma(l-\chi_j)(\psi(l-\chi_j)+\gamma) + L\Gamma(-\chi_j)(\psi(-\chi_j)+\gamma) \\
&\quad + \Xi - \frac{1}{12}\Big(18\Psi(1,-\chi_j) + 18\big(\psi(-\chi_j)+\gamma\big)^2 + L^2 - \pi^2\Big)\Gamma(-\chi_j) \\
&= -L\sum_{l\geq 1} \frac{(-1)^l}{l!(2^l-1)}\Gamma(l-\chi_j)\big(\psi(l-\chi_j)+\gamma\big) \\
&\quad - L\sum_{l\geq 0} \frac{(-1)^l}{l!}\Gamma(l-\chi_j)\psi(l-\chi_j) - L\gamma\Gamma(-\chi_j) + L\Gamma(-\chi_j)(\psi(-\chi_j)+\gamma) \\
&\quad + \Xi - \frac{1}{12}\Big(18\Psi(1,-\chi_j) + 18\big(\psi(-\chi_j)+\gamma\big)^2 + L^2 - \pi^2\Big)\Gamma(-\chi_j).
\end{aligned}$$

One sum must be interpreted as follows:

$$\begin{aligned}\sum_{l\ge0}\frac{(-1)^l}{l!}\Gamma(l-\chi_j)\psi(l-\chi_j) &= \sum_{l\ge0}\frac{(-1)^l}{l!}\frac{d}{dx}\Gamma(x+l-\chi_j)\Big|_{x=0}\\ &= \frac{d}{dx}\sum_{l\ge0}\frac{(-1)^l}{l!}\Gamma(x+l-\chi_j)\Big|_{x=0}\\ &= \frac{d}{dx}\Gamma(x-\chi_j)\sum_{l\ge0}\binom{-x+\chi_j}{l}\Big|_{x=0}\\ &= \frac{d}{dx}\Gamma(x-\chi_j)2^{-x}\Big|_{x=0}\\ &= \Gamma(-\chi_j)\big(\psi(-\chi_j)-L\big).\end{aligned}$$

So

$$\begin{aligned}I &= -L\sum_{l\ge1}\frac{(-1)^l}{l!(2^l-1)}\Gamma(l-\chi_j)\big(\psi(l-\chi_j)+\gamma\big)\\ &\quad+\Xi-\frac{1}{12}\Big(18\Psi(1,-\chi_j)+18\big(\psi(-\chi_j)+\gamma\big)^2-11L^2-\pi^2\Big)\Gamma(-\chi_j).\end{aligned}$$

But also

$$\begin{aligned}I &= \frac{1}{2\pi i}\int_{(1/2)}\frac{L}{2^z-1}\Gamma(-z)\big(\psi(-z)+\gamma\big)\Gamma(z-\chi_j)dz\\ &= L\sum_{l\ge1}\frac{(-1)^l\Gamma(l-\chi_j)}{l!}\Big[\frac{\psi(l-\chi_j)+\gamma}{2^l-1}-\frac{2^lL}{(2^l-1)^2}\Big].\end{aligned}$$

Combining these two expressions for I we find

$$\begin{aligned}\Xi &= 2L\sum_{l\ge1}\frac{(-1)^l}{l!(2^l-1)}\Gamma(l-\chi_j)\big(\psi(l-\chi_j)+\gamma\big)-L^2\sum_{l\ge1}\frac{(-1)^l\Gamma(l-\chi_j)}{l!}\frac{2^l}{(2^l-1)^2}\\ &\quad+\frac{1}{12}\Big(18\Psi(1,-\chi_j)+18\big(\psi(-\chi_j)+\gamma\big)^2-11L^2-\pi^2\Big)\Gamma(-\chi_j).\end{aligned}$$

For our application, it gets a factor $2/L^3$, but now let us engage into the remaining term:

$$\spadesuit=\sum_{k\ne0,\ne j}\Gamma(l-\chi_k)\Gamma(\chi_k-\chi_j).$$

Set

$$F=\frac{L}{2^z-1}\Gamma(l-z)\Gamma(z-\chi_j),$$

then

$$
\begin{aligned}
I &= \frac{1}{2\pi i}\int_{(1/2)} \frac{L}{2^z-1}\Gamma(l-z)\Gamma(z-\chi_j)dz \\
&= \frac{1}{2\pi i}\int_{(-1/2)} \frac{L}{2^z-1}\Gamma(l-z)\Gamma(z-\chi_j)dz \\
&\quad + \spadesuit + (l-1)!\Gamma(-\chi_j) - \Gamma(-\chi_j)\Big(\psi(-\chi_j)+\gamma+\frac{L}{2}\Big) \\
&= -\frac{L}{2\pi i}\int_{(-1/2)} \frac{1}{2^{-z}-1}\Gamma(l-z)\Gamma(z-\chi_j)dz \\
&\quad - \frac{L}{2\pi i}\int_{(-1/2)} \Gamma(l-z)\Gamma(z-\chi_j)dz \\
&\quad + \spadesuit + (l-1)!\Gamma(-\chi_j) - \Gamma(-\chi_j)\Big(\psi(-\chi_j)+\gamma+\frac{L}{2}\Big) \\
&= -\frac{L}{2\pi i}\int_{(-1/2)} \frac{1}{2^{-z}-1}\Gamma(l-\chi_j-z)\Gamma(z)dz \\
&\quad - \frac{L}{2\pi i}\int_{(-1/2)} \Gamma(l-z)\Gamma(z-\chi_j)dz \\
&\quad + \spadesuit + (l-1)!\Gamma(-\chi_j) - \Gamma(-\chi_j)\Big(\psi(-\chi_j)+\gamma+\frac{L}{2}\Big) \\
&= -L\sum_{h\geq 1}\Gamma(l+h-\chi_j)\frac{(-1)^h}{h!(2^h-1)} \\
&\quad - L\sum_{h\geq l}\frac{(-1)^{h-l}}{(h-l)!}\Gamma(h-\chi_j) + L\Gamma(L-\chi_j) \\
&\quad + \spadesuit + (l-1)!\Gamma(-\chi_j) - \Gamma(-\chi_j)\Big(\psi(-\chi_j)+\gamma+\frac{L}{2}\Big) \\
&= -L\sum_{h\geq 1}\Gamma(l+h-\chi_j)\frac{(-1)^h}{h!(2^h-1)} \\
&\quad - L\Gamma(l-\chi_j)2^{-l} + L\Gamma(l-\chi_j) \\
&\quad + \spadesuit + (l-1)!\Gamma(-\chi_j) - \Gamma(-\chi_j)\Big(\psi(-\chi_j)+\gamma+\frac{L}{2}\Big).
\end{aligned}
$$

But also

$$
\begin{aligned}
I &= \frac{1}{2\pi i}\int_{(1/2)} \frac{L}{2^z-1}\Gamma(l-z)\Gamma(z-\chi_j)dz \\
&= L\sum_{h\geq l}\frac{(-1)^{h-l}}{(h-l)!(2^h-1)}\Gamma(h-\chi_j) \\
&= L\sum_{h\geq 0}\frac{(-1)^h}{h!(2^{h+l}-1)}\Gamma(h+l-\chi_j).
\end{aligned}
$$

So

$$
\begin{aligned}
&- L\sum_{h\geq 1}\Gamma(l+h-\chi_j)\frac{(-1)^h}{h!(2^h-1)} - L\Gamma(l-\chi_j)2^{-l} + L\Gamma(l-\chi_j) \\
&\quad + \spadesuit + (l-1)!\Gamma(-\chi_j) - \Gamma(-\chi_j)\Big(\psi(-\chi_j)+\gamma+\frac{L}{2}\Big) \\
&= L\sum_{h\geq 0}\frac{(-1)^h}{h!(2^{h+l}-1)}\Gamma(h+l-\chi_j),
\end{aligned}
$$

or

$$\spadesuit = L\sum_{h\ge 0}\frac{(-1)^h}{h!(2^{h+l}-1)}\Gamma(h+l-\chi_j) + L\sum_{h\ge 1}\Gamma(l+h-\chi_j)\frac{(-1)^h}{h!(2^h-1)}$$
$$+ L\Gamma(l-\chi_j)2^{-l} - L\Gamma(l-\chi_j) - (l-1)!\Gamma(-\chi_j) + \Gamma(-\chi_j)\Big(\psi(-\chi_j)+\gamma+\frac{L}{2}\Big).$$

For our application it gets a factor

$$\frac{2}{L^2}\sum_{l\ge 1}\frac{(-1)^l}{l!(2^l-1)}$$

The grand summing up:

$$[\delta^3(x)]_j = \frac{2}{L^3}\Bigg[2L\sum_{l\ge 1}\frac{(-1)^l}{l!(2^l-1)}\Gamma(l-\chi_j)\big(\psi(l-\chi_j)+\gamma\big) - L^2\sum_{l\ge 1}\frac{(-1)^l\Gamma(l-\chi_j)}{l!}\frac{2^l}{(2^l-1)^2}$$
$$+\frac{1}{12}\Big(18\Psi(1,-\chi_j)+18\big(\psi(-\chi_j)+\gamma\big)^2-11L^2-\pi^2\Big)\Gamma(-\chi_j)\Bigg]$$
$$+\frac{2}{L^2}\sum_{l\ge 1}\frac{(-1)^l}{l!(2^l-1)}\Bigg[L\sum_{h\ge 0}\frac{(-1)^h}{h!(2^{h+l}-1)}\Gamma(h+l-\chi_j) + L\sum_{h\ge 1}\Gamma(l+h-\chi_j)\frac{(-1)^h}{h!(2^h-1)}$$
$$+ L\Gamma(l-\chi_j)2^{-l} - L\Gamma(l-\chi_j) - (l-1)!\Gamma(-\chi_j) + \Gamma(-\chi_j)\Big(\psi(-\chi_j)+\gamma+\frac{L}{2}\Big)\Bigg]$$
$$+\left[\frac{\pi^2}{6L^3}+\frac{1}{12L}-\frac{1}{L}-\frac{2}{L^2}\sum_{l\ge 1}\frac{(-1)^{l-1}}{l(2^l-1)}\right]\Gamma(-\chi_j).$$

We did not make an effort to simplify this ungainly expression any further.

5.2 The Adaptive Sampling algorithm.

We have, from (21),

$$F(r,\eta) = \sum_{l=0}^{\infty}\exp(-2^{-(\eta-l)})\frac{2^{-r(\eta-l)}}{r!}\left[1-\exp(-2^{-(\eta-l)})\sum_{k=0}^{b-r}\frac{2^{-k(\eta-l)}}{k!}\right].$$

We are interested in the moments of $Z := r2^{X_n}/n$ and $W := X_n - \log n$. So we first compute

$$\varphi(r,\alpha) = -\alpha\int_{-\infty}^{+\infty}e^{\alpha\eta}F(r,\eta)d\eta = -\alpha/(1-e^{\alpha})\left[\frac{\Gamma(r-\tilde{\alpha})}{Lr!} - \sum_{k=0}^{b-r}\frac{\Gamma(r+k-\tilde{\alpha})}{Lr!k!2^{r+k-\tilde{\alpha}}}\right],\ \Re(\alpha) < rL.$$

This leads to

$$\phi(r,\alpha) = \frac{\Gamma(r-\tilde{\alpha})}{Lr!} - \sum_{k=0}^{b-r}\frac{\Gamma(r+k-\tilde{\alpha})}{Lr!k!2^{r+k-\tilde{\alpha}}}.$$

The fundamental strip for (6) is $\Re(s)\in\langle -r, 0\rangle$.

5.2.1 Atom at $Z = 0$.

We have (the RV R is defined in Section 4.2)

$$\Pi_0 := \Pr(Z=0) = \Pr(R=0).$$

This leads to

$$\Pi_0 = \phi(0,0) + w_0,$$

with

$$\phi(0,0) = \lim_{\alpha=0} \phi(0,\alpha) = 1 - \sum_{k=1}^{b} \frac{1}{Lk2^k},$$

which is already computed in Louchard [20]. Furthermore,

$$w_0 = \sum_{l\neq 0} \lim_{\alpha\to -L\chi_l} \phi(0,\alpha) e^{-2l\pi i \log n} = -\sum_{l\neq 0}\sum_{k=1}^{b} \frac{\Gamma(k+\chi_l)}{Lk!2^{k+\chi_l}} e^{-2l\pi i \log n}.$$

5.2.2

For Z, we must have $d \le b$. Using limits (indeed, we must take care of singularities when $r \le d$), (15) and (16) lead to

$$\begin{aligned} \tilde{m}_{1,d} &= \sum_{r=1}^{b} r^d \lim_{\alpha\to Ld} \phi(r,\alpha), \\ w_{1,d} &= \sum_{r=1}^{b}\sum_{l\neq 0} r^d \lim_{\alpha\to L(d-\chi_l)} \phi(r,\alpha) e^{-2l\pi i \log n}. \end{aligned}$$

Note that $w_{1,1} = 0$; $\tilde{m}_{1,d}, w_{1,d}$ are given, for $d = 1, 2$ in Flajolet [4]. The dominant part $\tilde{m}_{1,d}$ was already computed in Louchard [20]:

$$\tilde{m}_{1,d} = 1 + \frac{1}{L}\sum_{i=1}^{d} \frac{i^d(-1)^{d-i}B(d-i)}{i!(d-i)!} + \frac{1}{L}\sum_{k=1}^{d-1} \frac{\left\{{d \atop k}\right\}}{d-k}\left[\frac{1}{(d-k)!} - \frac{(b-d)!}{(b-k)!}\right](1-2^{d-k}),$$

with

$$B(k) := \sum_{j=1}^{k} \frac{(-1)^j}{j}.$$

Now we can use (4) and (13).

To simplify $\tilde{m}_{1,d}$, we rewrite one of the sums a bit:

$$\begin{aligned} \mathsf{SUM} &= \sum_{i=1}^{d} \frac{i^d(-1)^{d-i}B(d-i)}{i!(d-i)!} = \sum_{i=0}^{d-1} \frac{(d-i)^d(-1)^i}{i!(d-i)!} \sum_{1\le j\le i} \frac{(-1)^j}{j} \\ &= \sum_{\lambda=0}^{d} \left\{{d \atop \lambda}\right\} \sum_{i=0}^{d-\lambda} \frac{(d-i)!(-1)^i}{(d-i-\lambda)!i!(d-i)!} \sum_{1\le j\le i} \frac{(-1)^j}{j} = \sum_{\lambda=0}^{d} \left\{{d \atop \lambda}\right\} \sum_{i=0}^{d-\lambda} \frac{(-1)^i}{(d-i-\lambda)!i!} \sum_{1\le j\le i} \frac{(-1)^j}{j} \\ &= \sum_{\lambda=0}^{d} \left\{{d \atop \lambda}\right\} \sum_{1\le j\le d-\lambda} \frac{(-1)^j}{j} \frac{1}{(d-\lambda)!} \sum_{i=j}^{d-\lambda} (-1)^i \binom{d-\lambda}{i} \\ &= \sum_{\lambda=0}^{d-1} \left\{{d \atop \lambda}\right\} \sum_{1\le j\le d-\lambda} \frac{(-1)^j}{j} \frac{1}{(d-\lambda)!} (-1)^j \binom{d-\lambda-1}{j-1} \\ &= \sum_{\lambda=0}^{d-1} \left\{{d \atop \lambda}\right\} \frac{1}{(d-\lambda)(d-\lambda)!} \sum_{1\le j<d} \binom{d-\lambda}{j} = \sum_{\lambda=0}^{d-1} \left\{{d \atop \lambda}\right\} \frac{2^{d-\lambda}-1}{(d-\lambda)(d-\lambda)!}. \end{aligned}$$

Then

$$\begin{aligned} \tilde{m}_{1,d} &= 1 + \frac{1}{L}\sum_{k=1}^{d-1} \left\{{d \atop k}\right\} \frac{2^{d-k}-1}{(d-k)(d-k)!} + \frac{1}{L}\sum_{k=1}^{d-1} \frac{\left\{{d \atop k}\right\}}{d-k}\left[\frac{1}{(d-k)!} - \frac{(b-d)!}{(b-k)!}\right](1-2^{d-k}) \\ &= 1 + \frac{(b-d)!}{L}\sum_{k=1}^{d-1} \left\{{d \atop k}\right\} \frac{2^{d-k}-1}{(d-k)(b-k)!}. \end{aligned}$$

(The extra term 1 corresponds to the index $k = d$, if we interpret it as a limit.)

Finally, we can also work out an explicit form for $w_{1,d}$:

$$w_{1,d} = \sum_{l\neq 0}\sum_{r=1}^{b} r^d \lim_{\alpha\to d-\chi_l}\left[\frac{\Gamma(r-\alpha)}{Lr!} - \sum_{k=0}^{b-r}\frac{\Gamma(r+k-\alpha)}{Lr!k!2^{r+k-\alpha}}\right]e^{-2l\pi i\log n}.$$

It is nicer to work with falling factorials instead of powers, which will bring in Stirling numbers at the very end. So we look at (for $j \le d$):

$$\begin{aligned}
S &= \sum_{r=j}^{b} r^{\underline{j}}\left[\frac{\Gamma(r-d+\chi_l)}{Lr!} - \sum_{k=0}^{b-r}\frac{\Gamma(r+k-d+\chi_l)}{Lr!k!2^{r+k-d}}\right]\\
&= \frac{1}{L}\sum_{r=j}^{b}\left[\frac{\Gamma(r-d+\chi_l)}{(r-j)!} - \sum_{k=0}^{b-r}\frac{\Gamma(r+k-d+\chi_l)}{(r-j)!k!2^{r+k-d}}\right]\\
&= \frac{1}{L}\sum_{r=0}^{b-j}\frac{\Gamma(r+j-d+\chi_l)}{r!} - \frac{1}{L}\sum_{r=0}^{b-j}\sum_{k=0}^{b-j-r}\frac{\Gamma(r+k+j-d+\chi_l)}{r!k!2^{r+k+j-d}}\\
&= \frac{1}{L}\sum_{r=0}^{b-j}\frac{\Gamma(r+j-d+\chi_l)}{r!} - \frac{1}{L2^{j-d}}\sum_{h=0}^{b-j}\sum_{k=0}^{h}\frac{\Gamma(h+j-d+\chi_l)}{(h-k)!k!2^{h}}\\
&= \frac{1}{L}\sum_{r=0}^{b-j}\frac{\Gamma(r+j-d+\chi_l)}{r!} - \frac{1}{L2^{j-d}}\sum_{h=0}^{b-j}\frac{\Gamma(h+j-d+\chi_l)}{h!}\\
&= \frac{1}{L}\Big(1-\frac{1}{2^{j-d}}\Big)\Gamma(j-d+\chi_l)\sum_{r=0}^{b-j}\binom{r+j-1-d+\chi_l}{r}\\
&= \frac{1}{L}\Big(1-\frac{1}{2^{j-d}}\Big)\Gamma(j-d+\chi_l)\binom{b-d+\chi_l}{b-j}
\end{aligned}$$

and find eventually

$$w_{1,d} = \sum_{l\neq 0}\frac{1}{L}\sum_{j=1}^{d-1}\left\{{d \atop j}\right\}\Big(1-\frac{1}{2^{j-d}}\Big)\Gamma(j-d+\chi_l)\binom{b-d+\chi_l}{b-j}e^{-2l\pi i\log n}.$$

5.2.3

For W, (3) and (10) lead to

$$\begin{aligned}
\tilde{m}_j &= \sum_{r=0}^{b}\phi^{(j)}(r,0),\\
w_j &= \sum_{r=0}^{b}\sum_{l\neq 0}\phi^{(j)}(r,\alpha)\Big|_{\alpha=-L\chi_l}e^{-2l\pi i\log n}.
\end{aligned}$$

This gives

$$\begin{aligned}
\tilde{m}_1 &= 3/2 + (\gamma - L)/L + \sum_{k=1}^{b} \frac{\psi(k) - L}{L^2 k 2^k} + \sum_{r=1}^{b} \left[-\frac{1}{Lr2^r} + \frac{\Psi(r)(2^{-r}-1)}{L^2 r} \right] \\
&+ \sum_{r=1}^{b} \sum_{k=1}^{b-r} \frac{-[L - \Psi(r+k)]2^{-(r+k)}\Gamma(r+k)}{L^2\Gamma(k+1)\Gamma(r+1)}, \\
\tilde{m}_2 &= 1/3 + \gamma/L + \gamma^2/L^2 + \pi^2/(6L^2) \\
&+ \sum_{r=1}^{b} \left[-\frac{1}{Lr2^r} + \frac{-\Psi(1,r)(2^{-r}-1)}{L^3 r} + \frac{\Psi(r)[\Psi(r)(1-2^{-r}) + 2^{1-r}L]}{L^3 r} \right] \\
&+ \sum_{k=1}^{b} \frac{-\Psi(1,k) - \Psi(k)^2 + 2L\Psi(k)}{L^3 k 2^k} \\
&- \sum_{r=1}^{b} \sum_{k=1}^{b-r} \frac{[-2\Psi(r+k)L + L^2 + \Psi(1,r+k) + \Psi(r+k)^2]2^{-(r+k)}\Gamma(r+k)}{L^3\Gamma(k+1)\Gamma(r+1)}, \\
w_1 &= \sum_{r=0}^{b} \sum_{l\neq 0} \frac{[-L2^{-r} + \Psi(r+\chi_l)(2^{-r}-1)]\Gamma(r+\chi_l)}{L^2\Gamma(r+1)} e^{-2l\pi i \log n} \\
&+ \sum_{r=0}^{b} \sum_{k=1}^{b-r} \sum_{l\neq 0} \frac{2^{-(r+k)}[-L + \Psi(r+k+\chi_l)]\Gamma(r+k+\chi_l)}{L^2\Gamma(r+1)\Gamma(k+1)} e^{-2l\pi i \log n} \\
w_2 &= \sum_{l\neq 0} \left[\sum_{r=0}^{b} \frac{[-L^2 2^{-r} - \Psi(1,r+\chi_l)(2^{-r}-1) - \Psi(r+\chi_l)^2(2^{-r}-1) + \Psi(r+\chi_l)L2^{1-r}]\Gamma(r+\chi_l)}{L^3\Gamma(r+1)} \right. \\
&\left. - \sum_{r=0}^{b} \sum_{k=1}^{b-r} \frac{[L^2 - 2\Psi(r+k+\chi_l) + \Psi(1,r+k+\chi_l) + \Psi(r+k+\chi_l)^2]2^{-(r+k)}\Gamma(r+k+\chi_l)}{L^2\Gamma(r+1)\Gamma(k+1)} \right] e^{-2l\pi i \log n}.
\end{aligned}$$

Moments of order 3 are rather long, so we don't give them here (but they are computed with Maple).

Now we can use (4) and (13).

5.3 LogLog Counting.

We obtain, with (3),

$$\phi(\alpha) = \left[\Gamma(1 - \tilde{\alpha}/m) \frac{e^{\alpha/m} - 1}{\alpha/m} \right]^m, \ \Re(\alpha) < mL.$$

With (15), we must have $d < m$. The fundamental strip for s is $\Re(s) \in \langle -m, 0 \rangle$. Equation (15) gives

$$\tilde{m}_{1,k} = H_3(k),$$

with

$$H_3(k) := \left[\Gamma(-k/m)(1 - e^{kL/m})/L \right]^m.$$

Equation (17) gives

$$\begin{aligned}
\tilde{\mu}_{1,2} &= -H_3(1)^2 + H_3(2), \\
\tilde{\mu}_{1,3} &= 2H_3(1)^3 + H_3(3) - 3H_3(2)H_3(1).
\end{aligned}$$

Equation (16) gives

$$w_{1,1} = \sum_{l\neq 0} H_4(1, \chi_l) e^{-2l\pi i \log(n/m)},$$

with

$$H_4(k,\chi_l) := \left[\Gamma((\chi_l - k)/m)(1 - e^{L(k-\chi_l)/m})/L\right]^m.$$

$\tilde{m}_{1,1}$, $\tilde{\mu}_{1,2}$ are given in Durand, Flajolet [2].

Equation (18) gives

$$\begin{aligned}
\kappa_{1,2} &= -2H_3(1)w_{1,1} - w_{1,1}^2 + \sum_{l\neq 0} H_4(2,\chi_l)e^{-2l\pi i \log(n/m)},\\
\kappa_{1,3} &= 6H_3(1)^2 w_{1,1} + 6H_3(1)w_{1,1}^2 - 3H_2(2)w_{1,1} + 2w_{1,1}^3\\
&+ \sum_{l\neq 0}\{-3H_4(2,\chi_l)H_3(1) - 3H_4(2,\chi_l)w_{1,1} + H_4(3,\chi_l)\}e^{-2l\pi i \log(n/m)}.
\end{aligned}$$

5.4 Distinctness of Compositions of an integer.

The approach is different here. From Hwang and Yeh [9], Hitczenko and Louchard [8], we know that

$$\mathbb{E}(X_n) = \log n - 1/2 + \gamma/L + w_{1,T} + \mathcal{O}(1/n),$$

where $w_{1,T}$ is the Trie's w_1, Equation (25). Moreover, setting, as in the Trie case,

$$P(j) := (1 - 1/2^j)^n,$$

we also derive from [8], with asymptotic independence of the urns representing each integer,

$$\mathbb{E}(e^{\alpha X_n}) = \exp\left[\sum_{j=1}^{\infty} \ln\left(1 + (e^\alpha - 1)(1 - P(j))\right)\right] = \exp\left[\sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l}(e^\alpha - 1)^l V_l\right],$$

with

$$V_l := \sum_{j=1}^{\infty}(1 - P(j))^l.$$

We obtain

$$\begin{aligned}
V_l &= \sum_{j=1}^{\infty}\left\{\sum_{k=0}^{l}(-1)^k\binom{l}{k}P(j)^k\right\}\\
&= \sum_{j=1}^{\infty}\left\{\sum_{k=0}^{l}(-1)^k\binom{l}{k}P(j)^k - \sum_{k=0}^{l}\binom{l}{k}(-1)^k\right\}\\
&= \sum_{k=1}^{l}\binom{l}{k}(-1)^{k+1}S_k, \text{ with}\\
S_k &:= \sum_{j=1}^{\infty}(1 - P(j)^k).
\end{aligned}$$

Setting $\eta = j - \log(nk)$, this leads to

$$P(j)^k \sim F(\eta) = \exp(-2^{-\eta}),$$

and S_k is the mean of the RV with distribution function $P(j)^k$, minus 1 (as the sum starts here at $j = 1$). From Section 5.1, we immediately obtain ($\tilde{m}_{1,T}$ denotes here the Tries' $\tilde{m}_1$ and similarly for $w_{1,T}$)

$$S_k = \tilde{m}_{1,T} - 1 + \log(nk) + \beta_{1,k} + \mathcal{O}(1/n),$$

with

$$\begin{aligned}\tilde{m}_{1,T} - 1 &= -1/2 + \gamma/L,\\ \beta_{1,k} &= -\sum_{l\neq 0} \Gamma(\chi_l) e^{-2l\pi i \log(nk)}/L.\end{aligned}$$

Note that $\beta_{1,4} = \beta_{1,2} = \beta_{1,1} = w_{1,T}$. This gives

$$\begin{aligned}V_l &= \log n - 1/2 + \gamma/L + B_l + \beta_l, \text{ with}\\ B_l &:= \sum_{k=2}^{l} \binom{l}{k} (-1)^{k+1} \log k,\\ \beta_l &= \sum_{k=1}^{l} \binom{l}{k} (-1)^{k+1} \beta_{1,k},\\ \beta_1 &= \beta_{1,1} = w_{1,T},\end{aligned}$$

and, finally

$$\begin{aligned}&\phi_p(\alpha) e^{\alpha \log n} = \mathbb{E}(e^{\alpha X_n})\\ &= \exp\left[\alpha(\log n - 1/2 + \gamma/L) + \sum_{l=2}^{\infty} \frac{(-1)^{l+1}}{l} (e^\alpha - 1)^l B_l + \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} (e^\alpha - 1)^l \beta_l + \mathcal{O}(1/n)\right].\end{aligned}$$

From this, we derive

$$\begin{aligned}\tilde{m}_1 &= \gamma/L - 1/2,\\ \tilde{m}_2 &= 5/4 - \gamma/L + \gamma^2/L^2,\\ \tilde{\mu}_2 &= 1,\\ \tilde{\mu}_3 &= -3 + 2\log 3,\\ \kappa_2 &= 0,\\ \kappa_3 &= -2w_1 + 2\beta_{1,3}.\end{aligned}$$

Note that the variance has no periodic component; $\tilde{m}_l$, $\tilde{\mu}_l$ were given in [8].

For interest, we report about a direct approach, that requires to prove directly that the periodic components vanish. This is taken from [27].

The variance of the insertion cost of a random Patricia trie constructed from n random data was computed in [13] as

$$\frac{\pi^2}{6L^2} + \frac{1}{12} - \frac{2}{L} \sum_{l\geq 1} \frac{(-1)^{l-1}}{l(2^l - 1)} + \sigma_V(\log_2 n) + o(1),$$

with[3]

$$\begin{aligned}\sigma_V(x) = &-\frac{2}{L^2} \sum_{k\neq 0} (1+\chi_k)\Gamma(-1-\chi_k)\big(\psi(-1-\chi_k) + \gamma\big) e^{2\pi i k x}\\ &- \frac{2}{L} \sum_{k\neq 0} (1+\chi_k)\Gamma(-1-\chi_k) g(\chi_k) e^{2\pi i k x} - \big(\delta_E(x)\big)^2,\end{aligned}$$

[3]There is a typo; the paper [12] contains the correct version. This paper discusses M-ary Patricia tries, and only for the instance $M = 2$ (binary Patricia tries) do the cancellations occur.

and

$$\delta_E(x) = \frac{1}{L}\sum_{k\neq 0}(1+\chi_k)\Gamma(-1-\chi_k)e^{2\pi ikx},$$
$$g(x) := \sum_{l\geq 1}\binom{x}{l}\frac{1}{2^l-1}.$$

For interest, the variance is computed from the recursion

$$K_n(u) = 2^{1-n}u\sum_{k=0}^{n}\binom{n}{k}K_k(u) - 2^{1-n}(u-1)K_n(u),\ n\geq 2, \quad K_0(u) = K_1(u) = 1$$

via

$$\frac{K_n{}''(1)}{n} + \frac{K_n{}'(1)}{n} - \left(\frac{K_n{}'(1)}{n}\right)^2.$$

These results can be rewritten, using $\Gamma(x+1) = x\Gamma(x)$ and $\psi(x+1) = \psi(x) + \frac{1}{x}$:

$$\sigma_V(x) = \frac{2}{L^2}\sum_{k\neq 0}\Gamma(-\chi_k)\big(\psi(-\chi_k)+\gamma\big)e^{2\pi ikx}$$
$$+ \frac{2}{L}\sum_{k\neq 0}\Gamma(-\chi_k)g(\chi_k)e^{2\pi ikx} - \big(\delta_E(x)\big)^2,$$

and

$$\delta_E(x) = -\frac{1}{L}\sum_{k\neq 0}\Gamma(-\chi_k)e^{2\pi ikx}.$$

Now if we look at the coefficient of $e^{2\pi ikx}$, for $k\neq 0$, we find

$$\frac{2}{L^2}\Gamma(-\chi_k)\big(\psi(-\chi_k)+\gamma\big) - \frac{2}{L}\Gamma(-\chi_k)g(\chi_k) - \frac{1}{L^2}\sum_{j\neq 0,\neq k}\Gamma(-\chi_j)\Gamma(-\chi_{k-j}).$$

According to the analysis in the previous sections, these coefficients are all equal to zero.

Patricia tries offered a surprise in 1986 when it turned out that the constant

$$\frac{\pi^2}{6L^2} + \frac{1}{12} - \frac{2}{L}\sum_{l\geq 1}\frac{(-1)^{l-1}}{l(2^l-1)}$$

is just $1.000000000001237\ldots$. That was nicely explained by Johannes Schoißengeier [14].

A similar phenomenon [27] occurs for words $x_1\ldots x_n$, where the letters follow (independent) geometric random variables X with $\Pr\{X=i\} = 2^{-i}$ for $i = 1,2,3,\ldots$. Then the variance of the parameter "number of different letters in a random word of length n" is $1+o(1)$. The cancellation is equivalent to the cancellation in the Patricia instance.

5.5 The Probabilistic Counting and first empty part in Compositions of integers.

We have, from (24),

$$F(\eta) = 1 - \prod_{k=0}^{\infty}[1-\exp(-e^{-L(\eta-k)})].$$

But as

$$\sum_{j=0}^{\infty}(-1)^{\nu(j)}q^j = \prod_{k=0}^{\infty}(1-q^{2^k}),$$

we obtain

$$F(\eta) = 1 - \sum_{l=0}^{\infty} (-1)^{\nu(l)} \exp(-le^{-L\eta}),$$

which is the expression Flajolet and Martin used in [6]. This leads to

$$\varphi(\alpha) = -N(-\tilde{\alpha})\Gamma(1-\tilde{\alpha}),\ \Re(\alpha) < L.$$

where $N(s)$ is the analytic continuation of

$$\sum_{j\geq 1} (-1)^{\nu(j)}/j^s.$$

We will need the values of the derivatives of $N(s)$. As shown in [6], we can group the terms of $N(s)$ together. For instance, we can use the fact that

$$\begin{aligned}
\nu(8j) &= \nu(j), \\
\nu(8j+1) &= \nu(8j+2) = \nu(8j+4) = \nu(j)+1, \\
\nu(8j+3) &= \nu(8j+5) = \nu(8j+6) = \nu(j)+2, \\
\nu(8j+7) &= \nu(j)+3.
\end{aligned}$$

We obtain

$$\begin{aligned}
N(s) &= -1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} + 6^{-s} - 7^{-s} \\
&+ \sum_{j=1}^{\infty} \frac{(-1)^{\nu(j)}}{(8j)^s} \left[1 - \frac{1}{(1+1/(8j))^s} - \frac{1}{(1+2/(8j))^s} + \frac{1}{(1+3/(8j))^s} \right. \\
&- \left. \frac{1}{(1+4/(8j))^s} + \frac{1}{(1+5/(8j))^s} + \frac{1}{(1+6/(8j))^s} - \frac{1}{(1+7/(8j))^s} \right].
\end{aligned}$$

Differentiating the sum in $N(s)$ and setting $s = 0$ leads for instance to

$$\sum_{j=1}^{\infty} (-1)^{\nu(j)} \ln \left(\frac{(8j+1)(8j+2)(8j+4)(8j+7)}{8j(8j+3)(8j+5)(8j+6)} \right).$$

This leads to

$$\begin{aligned}
N(0) &= -1, \\
N'(0) &= -.4874506\ldots, \\
N''(0) &= .8433214\ldots, \\
N'''(0) &= -.8683385\ldots
\end{aligned}$$

5.5.1

Let us first analyze $X_n - \log n$. We derive

$$\begin{aligned}
m_1 &= (\gamma + N'(0))/L, \\
m_2 &= (\pi^2/6 + \gamma^2 + 2N'(0)\gamma - N''(0))/L^2, \\
m_3 &= (2\zeta(3) + \pi^2\gamma/2 + \gamma^3 + N'(0)\pi^2/2 + 3N'(0)\gamma^2 - 3N''(0)\gamma + N'''(0))/L^3, \\
\tilde{m}_1 &= (\gamma + N'(0))/L + 1/2, \\
\tilde{m}_2 &= (\gamma + N'(0))/L + 1/3 + (\pi^2/6 + \gamma^2 + 2N'(0)\gamma - N''(0))/L^2, \\
\sigma^2 &= (-N'(0)^2 + \pi^2/6 - N''(0))/L^2, \\
\mu_3 &= (2N'(0)^3 + 3N''(0)N'(0) + N'''(0) + 2\zeta(3)/L^3, \\
\tilde{\mu}_2 &= (-6N'(0)^2 + \pi^2 - 6N''(0))/(6L^2) + 1/12, \\
\tilde{\mu}_3 &= (2N'(0)^3 + 3N''(0)N'(0) + N'''(0) + 2\zeta(3))/L^3.
\end{aligned}$$

Let us now turn to the fluctuating components. The fundamental strip for (6) is $\Re(s) \in \langle -1, 0\rangle$. First of all, (7) and (9) lead to

$$w_1 = \sum_{l\neq 0} N(\chi_l)\Gamma(\chi_l)e^{-2l\pi i \log n}/L.$$

Equations (11) and (12) lead to

$$\kappa_2 = -w_1^2 - 2\sum_{l\neq 0}[N(\chi_l)(\gamma + N'(0)) + N'(\chi_l) + N(\chi_l)\psi(\chi_l)]\Gamma(\chi_l)e^{-2l\pi i \log n}/L^2.$$

Next we obtain

$$\begin{aligned}
\kappa_3 &= \sum_{l\neq 0}\big\{3\psi(1,\chi_l)N(\chi_l)/L^3 + \psi(\chi_l)[6N(\chi_l)w_1/L^2 + 6N'(\chi_l)/L^3 + 6N(\chi_l)(\gamma + N'(0))/L^3] \\
&+ 3N(\chi_l)\psi^2(\chi_l)/L^3 + 3(2(N'(0)+\gamma)N'(\chi_l) + N''(\chi_l))/L^3\big\}\,\Gamma(\chi_l)e^{-2l\pi i\log n} \\
&+ 6w_1\sum_{l\neq 0} N'(\chi_l)\Gamma(\chi_l)e^{-2l\pi i\log n}/L^2 \\
&- 1/2w_1(6\gamma^2 + 12\gamma N'(0) - 6N''(0) + L^2 + \pi^2)/L^2 - 6w_1^2(\gamma + N'(0))/L - 4w_1^3.
\end{aligned}$$

$\tilde{m}_1, \tilde{\mu}_2, w_1$ are given in [6] (with some discrepancies).

5.5.2

Now we analyze the moments $\mathbb{E}(\Delta^k)$ of $\Delta := m/n2^{\sum J_i/m}$ (m is again a design parameter). Equation (15) gives

$$\tilde{m}_{1,k} = H_5(k),$$

with

$$H_5(k) := \Big[N(-k/m)\Gamma(-k/m)(e^{kL/m} - 1)/L\Big]^m.$$

Equation (17) gives

$$\begin{aligned}
\tilde{\mu}_{1,2} &= -H_5(1)^2 + H_5(2), \\
\tilde{\mu}_{1,3} &= 2H_5(1)^3 + H_5(3) - 3H_5(2)H_3(1).
\end{aligned}$$

Equation (16) gives

$$w_{1,1} = \sum_{l\neq 0} H_6(1,\chi_l)e^{-2l\pi i\log(n/m)},$$

with

$$H_6(k,\chi_l) := \Big[N((\chi_l - k)/m)\Gamma((\chi_l - k)/m)(e^{L(k-\chi_l)/m} - 1)/L\Big]^m.$$

$\tilde{m}_{1,1}$, $\tilde{m}_{1,2}$ are given in Flajolet and Martin [6].

Equation (18) gives

$$\begin{aligned}
\kappa_{1,2} &= -2H_5(1)w_{1,1} - w_{1,1}^2 + \sum_{l\neq 0} H_6(2,\chi_l)e^{-2l\pi i\log(n/m)}, \\
\kappa_{1,3} &= 6H_5(1)^2 w_{1,1} + 6H_5(1)w_{1,1}^2 - 3H_5(2)w_{1,1} + 2w_{1,1}^3 \\
&+ \sum_{l\neq 0}\{-3H_6(2,\chi_l)H_5(1) - 3H_6(2,\chi_l)w_{1,1} + H_6(3,\chi_l)\}e^{-2l\pi i\log(n/m)}.
\end{aligned}$$

We see the complete analogy with LogLog Counting.

6 Conclusion.

Identifying the Gumbel distribution (or variants) in many problems from algorithms and discrete structures, an approximation at an early stage leads to simplified computations of moments. In this way, all moments can be (in principle) computed, in a fairly automatic fashion. This is in sharp contrast to many earlier papers, where even the expectation and variance are a challenge, as long as exact (and ungainly) expressions make the computations extremely heavy. Usually, there are periodic oscillations involved, and we demonstrate how to deal with them as well.

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